

MAXIMALLY SUPERSYMMETRIC THEORIES IN
LIGHT-CONE SUPERSPACE

By

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To my parents and my sister, for all their love and support.

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Reduced supersymmetric field theories retain a great deal of information regarding their higher-dimensional progenitors. In this dissertation, we describe how the $(\mathcal{N} = 4, d = 4)$ SuperYang-Mills theory may be “oxidized” into its parent theory, the fully ten-dimensional $\mathcal{N} = 1$ Yang-Mills. Remarkably, this is achieved by adding a single term to the four-dimensional transverse space derivatives. We work in light-cone superspace which is entirely free of auxiliary fields.

Very similar in structure to $(\mathcal{N} = 4, d = 4)$ Yang-Mills is the maximally supersymmetric $(\mathcal{N} = 8, d = 4)$ Supergravity. This theory is obtained by reduction from eleven-dimensional $\mathcal{N} = 1$ Supergravity. We show that this four-dimensional Supergravity theory may be restored to eleven dimensions in a very similar fashion.

CHAPTER 1 INTRODUCTION

The Standard Model of particle physics remains the towering achievement of the last half century. It describes all known interactions (probed by experiments), from the Hubble radius down to scales of order 10^{-16} cm. The $SU(3) \times SU(2) \times U(1)$ Standard Model describes three of the four fundamental interactions with a remarkable degree of precision. However the theory is not without its shortcomings. It fails to explain why such widely varying scales seem to occur in nature. Gravity, for example, operates at the Planck scale (10^{-33} cm) while the length scales involved in the Standard Model are around 10^{20} times larger. A natural question to ask is: why is the Standard Model consistent at a scale so much greater than the Planck scale?

Experimental evidence is also raising serious doubts regarding the completeness of the Standard Model in its present form. Flavor oscillations of neutrinos produced by cosmic rays offer evidence for neutrino masses (as do measurements of neutrino fluxes from the sun)—the Standard Model, however, has only massless neutrinos accompanying the charged leptons in β decays. The Standard Model also seems arbitrary in the sense that it involves 19 parameters (plus 10 for neutrino masses) such as gauge and Yukawa couplings of varying magnitudes. Observations also indicate the presence of a dominant amount of non-luminous matter in the universe, something the Standard Model does not explain. Thus, there are several indications that we must look beyond the Standard Model [1] for an accurate description of our universe.

Gravity along with the Standard Model describes all four interactions, one of which is mediated by the spin 2 graviton and the remaining three by $SU(3) \times SU(2) \times U(1)$ spin 1 gauge bosons. In addition, the theory involves the spin 0 Higgs, quarks, leptons and fifteen multiplets of spin $\frac{1}{2}$ fermions (in 3 generations of 5). The union of

gravity with quantum mechanics leads to a nonrenormalizable quantum field theory, an indication that new physics appears at very high energy. Thus, our current understanding of Nature and its interactions (as governed by the Standard Model+gravity) cannot be viewed as a fundamental picture but merely as fragments of a bigger theory. The aim is to determine the single theoretical structure that will explain all fundamental interactions and hopefully have few (if any) undetermined parameters.

Without affecting the consistency of a theory of gravity, the best method to smooth out its divergences, involves spreading out the gravitational interaction. This is string theory: In this theory, the graviton (and other elementary particles) are one-dimensional objects rather than point particles (as in quantum field theory). The benefits String Theory offers are enormous: to start with, every consistent string theory must contain a massless spin 2 particle whose interactions (at low energy) are general relativity—it also leads to a consistent theory of quantum gravity (perturbatively). In addition, String Theories lead to gauge groups that are large enough to include the Standard Model in them. String theories however suffer from too many symmetries and supersymmetries, and require ten space-time dimensions for consistency. Comparison with data requires an understanding, perhaps dynamical, of the breaking of these symmetries, for which we have no hint. The answer may lie in the eleven-dimensional M-theory [2]. Although very little is known about this theory, we do know that it subsumes all the known superstring theories and may contain guideposts for supersymmetry-breaking and dimensional reduction. Its infrared limit is the much studied $N = 1$ Supergravity in eleven dimensions [3], believed to be ultraviolet divergent.

Although M-theory casts well-defined shadows on lower-dimensional manifolds, its actual structure remains a mystery. Our hope therefore lies in unraveling its structure, starting from its known low energy limit, eleven-dimensional Supergravity. This $\mathcal{N} = 1$ theory in eleven dimensions is the largest supersymmetric local field theory

with maximal helicity two (on reduction to four dimensions). It is ultraviolet divergent and its divergences are presumably tamed by M-Theory. A clearer understanding of how this occurs will offer us a window into the workings of M-Theory.

1.1 Divergences and the Light-Cone

A technically difficult but conceptually simple framework for discussing divergences is the light-cone [4] frame formulation. Working in light-cone gauge is advantageous for two reasons. Primarily, all spurious degrees of freedom are eliminated and one deals exclusively with the physical degrees of freedom. Secondly, the role of the transverse little group [5] is made apparent. This opens up the possibility of relating the space-time group structure to the divergences that occur in a given theory.

Such a relation (between spacetime spin and divergences) was noticed by Richard Hughes [6] who pointed out that the coefficients of the one-loop β function in QCD were proportional to $(-1)^{2s}(1 - 12s^2)/3$ where s is the helicity of the circulating particle. This result was extended by Curtright [7] to dimensionally reduced theories. Curtright considered bosonic and fermionic contributions to the one loop vacuum polarization graph. He conjectured that these contributions (for a theory in d dimensions, obtained by reduction from D dimensions) was

$$(-1)^f \frac{1}{(16\pi^2)} g^2 f^{acd} f^{bcd} \frac{1}{\epsilon} \left[\frac{I^{(0)}}{d-1} (q^2 \delta_{\mu\nu}^{(d)} - q_\mu q_\nu) - 4 \frac{I^{(2)}}{r} (q^2 \delta_{\mu\nu}^{(D)} - q_\mu q_\nu) \right] \quad (1.1)$$

where f is 0 for bosons and 1 for fermions and the indices are those appropriate to either the boson or spinor representations of $SO(D-2)$. r is the rank of $SO(D-2)$ and q^μ , the usual four-momenta. This result is obtained, using supersymmetric dimensional regularization [8]. The p 'th Dynkin index is defined by

$$I^{(p)}[R] = \sum_{\mathbf{w} \text{ in } R} (\mathbf{w} \cdot \mathbf{w})^{\frac{p}{2}} \quad (1.2)$$

where $\mathbf{w}$ are the weight vectors in representation R . It follows that $I^{(0)}$ is simply the dimension of the representation. In 10 dimensions for example, a gauge boson has 8 components which means $I^{(0)}_{\text{boson}} = 8$.

A perfect check of this conjecture is the $\mathcal{N} = 4$ SuperYang-Mills theory. This theory is obtained by reducing ten-dimensional Yang-Mills where the relevant little group is $SO(8)$. Thanks to triality,

$$\begin{aligned} I^{(0)}_{\text{vector}} &= I^{(0)}_{\text{spinor}} = 8 \\ I^{(2)}_{\text{vector}} &= I^{(2)}_{\text{spinor}} = 8, \end{aligned}$$

and since fermions enter with an additional sign, finiteness of the $\mathcal{N} = 4$ theory at one loop follows. In fact triality also ensures that these indices match to all orders, offering an alternate proof of finiteness for the $\mathcal{N} = 4$ theory [9, 10].

Higher loop processes would then involve generalized higher order representation indices. In ten dimensions, the lack of divergences in string theories may be attributed to the triality of $SO(8)$, the light-cone little group. Applying a similar reasoning to eleven dimensions, we see that the incomplete cancelation of the Dynkin indices of the $SO(9)$ representations that describe $N = 1$ Supergravity is responsible for the divergences in that theory. Indeed, the mismatch between the eighth-order bosonic and fermionic Dynkin indices seems to support this conjecture (an eighth-order mismatch implies divergences for the $\mathcal{N} = 8$ Supergravity theory, at three loops or beyond).

The role of the little group is key to this analysis (and hence the need to work on the light-cone). Since M-Theory resides in $d = 11$, we expect special features of eleven-dimensional spacetime to be reflected in its physical little group, $SO(9)$. The first step is therefore to understand the divergent behavior of $\mathcal{N} = 1$ Supergravity. This requires a light-cone gauge description of the theory and will be one of the topics presented in this thesis.

Light-cone Supergravity actions (in component form) are simply too bulky to be useful. The sheer volume of terms at order κ and κ^2 makes it unsuitable for reading off Feynman rules (or for further calculations).

Superspace offers a compact alternative to this component approach. Superspace is an extension of ordinary spacetime, to include extra anticommuting coordinates in the form of $\mathcal{N}$ two-component Weyl spinors θ . We then define functions over this modified space, called superfields. These superfields may be expanded in a Taylor series, with respect to the anticommuting coordinates (the series is finite since the square of an anticommuting variable is zero). The coefficients obtained in this process are the usual component fields. Supersymmetry is manifest in superspace and its algebra is represented by translations and rotations, involving both commuting and anticommuting coordinates.

Light-cone superspace is the perfect venue to study supersymmetric theories. There is a complete lack of auxiliary fields, due to working on the light-cone and we will find that the actions are tractable even at higher orders in the coupling constant. In the next few chapters, we will describe the formulation of the ten and eleven-dimensional theories (discussed above), in light-cone superspace.

1.2 Overview

The four-dimensional $\mathcal{N} = 4$ SuperYang-Mills was formulated in light-cone superspace by Brink, Bengtsson and Bengtsson (and independently by Mandelstam) [12]. In reference [12] the authors introduced a single superfield that captured all the degrees of freedom of the $\mathcal{N} = 4$ theory. Due to the maximal supersymmetry in the theory, the superfield is extremely constrained. The action for the $\mathcal{N} = 4$ theory (in terms of this superfield) was obtained by requiring SuperPoincaré invariance in four dimensions.

This thesis assumes as a starting point their paper. We show that the above introduced $\mathcal{N} = 4$ superfield is sufficiently rich in structure to completely describe the fully ten-dimensional $\mathcal{N} = 1$ Yang-Mills. We achieve this by introducing a new “generalized” derivative. This involves extending the $d = 4$ transverse space derivatives into superspace. We prove that the derivative transforms appropriately under the SuperPoincaré generators and then show that simply generalizing the derivatives that appear in the four-dimensional action leads to the fully ten-dimensional theory.

($\mathcal{N} = 8, d = 4$) Supergravity was also formulated in light-cone superspace (up to first order in the gravitational coupling) by the same authors [11, 12]. Again, since the theory is maximally supersymmetric, its superfield contains sufficient information, to describe its higher-dimensional progenitor. Along similar lines to the Yang-Mills case, we introduce the missing coordinates and derivatives and show that this “ $d=4$ ” superfield may be used to completely describe the fully eleven-dimensional $\mathcal{N} = 1$ Supergravity.

We call this process dimensional Oxidation [13, 14] and explain in detail the various checks one may perform to verify its consistency. While the aim is to build light-cone actions for field theories with a view to analyzing their divergences, this oxidation process by itself is fascinating and warrants some discussion.

1.3 Dimensional Oxidation

Dimensional reduction [15] offers a mechanism to introduce new physics into a lower-dimensional universe [16]. It is, however, far more difficult (and often impossible) to start with a lower-dimensional theory and uniquely reconstruct its higher-dimensional parent. Reduced supersymmetric field theories are fairly unique in this regard. They tend to retain a great deal of information regarding their once higher-dimensional existence. This “memory” coded into their spectra allows for the unique recovery of their higher-dimensional progenitors.

A perfect candidate that displays such “memory” is the reduced maximally supersymmetric ($\mathcal{N} = 4, d = 4$) Yang-Mills theory. This theory is obtained by reduction from ($\mathcal{N} = 1, d = 10$) Yang-Mills. The light-cone little group decomposition relevant to this reduction is $SO(8) \supset SO(2) \times SO(6) \approx SO(2) \times SU(4)$. Signatures of ten dimensions are obvious in the $d = 4$ theory: the six scalar fields in the $\mathcal{N} = 4$ spectrum serve as reminders of a lost (compactified) $SO(6)$ while the $SU(4)$ spinors assemble into a single eight-spinor: $\mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2} = \mathbf{8}_s$ (of $SO(8)$). Also, the spectrum of the parent theory remains totally intact in the $d = 4$ version: for example, the $\mathbf{8}_v$ (in $d = 10$) is simply reinterpreted (in four dimensions) as, $\mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$. Thus, the spectrum of the $\mathcal{N} = 4$ theory hints at a ten-dimensional formulation (governed by the little group $SO(8)$). In principle, one simply needs to reintroduce the six missing coordinates (and their derivatives) to recover the $SO(8)$ -invariant theory.

Another equally impressive illustration of such “memory” is $\mathcal{N} = 8$ Supergravity in four dimensions. Starting from eleven-dimensional Supergravity, the little-group decomposition reads

$$SO(9) \supset SO(2) \times SO(7). \quad (1.3)$$

This theory is reminded of its origins because

- The $SO(9)$ structure is apparent in $\mathbf{8}_{1/2} + \bar{\mathbf{8}}_{-1/2} = \mathbf{16}_s$
- The $35 + \bar{35}$ scalars point to the, $\frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4}$ of $SO(7)$.

Again, obtaining the covariant ($\mathcal{N} = 1, d = 11$) Supergravity action should involve the introduction of the missing directions. Both the $\mathcal{N} = 4$ and $\mathcal{N} = 8$ theories are maximally supersymmetric - this produces the mirror-symmetry in their superfields (which one may argue, is the source of their “memory”). This “memory” is the key to our oxidation process. We will show that these theories are lifted perfectly into their once higher-dimensional avatars and the entire process is consistent.

CHAPTER 2 THE SUPERPOINCARÉ ALGEBRA

2.1 Supersymmetry

This chapter is a quick review of some relevant results from existing literature, pertaining to supersymmetry and specifically to the light-cone gauge. We will then use these results in subsequent chapters.

The known symmetries of the S-matrix in particle physics are Poincaré invariance, internal global symmetries (whose generators are Lorentz scalars) related to conserved quantum numbers like electric charge, and the discrete symmetries, C, P and T. Subject to certain assumptions, Coleman and Mandula [17] showed that these are the only possible symmetries. However, their statement assumed that the symmetry algebra of the S-matrix involved only commutators. Introducing anticommutators into the algebra leads to the possibility of supersymmetry. Supersymmetry [18–20], involves the introduction of anticommuting symmetry generators which transform in the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations of the Lorentz group. Since the generators are not scalars it is not an internal symmetry. The bosonic generators are, therefore, the four-momenta p_μ , the six Lorentz generators, $M_{\mu\nu}$ and a certain number of Hermitian internal symmetry generators, B_r . The algebra is that of the Poincaré group (in four dimensions, we set $\eta^{\mu\nu} = (-, +, +, +)$ with $\mu, \nu = 1 \dots 4$)

$$\begin{aligned}
 [p_\mu, p_\nu] &= 0 \\
 [p_\mu, M_{\rho\sigma}] &= i(\eta_{\mu\rho} p_\sigma - \eta_{\mu\sigma} p_\rho) \\
 [M_{\rho\sigma}, M_{\rho\sigma}] &= i(\eta_{\mu\rho} M_{\rho\sigma} - \eta_{\mu\rho} M_{\rho\sigma} - \eta_{\mu\rho} M_{\rho\sigma} + \eta_{\mu\rho} M_{\rho\sigma})
 \end{aligned}
 \tag{2.1}$$

together with that of the internal symmetry group (with indices $r, s \dots$)

$$[B_r, B_s] = if_{rs}^t B_t \quad (2.2)$$

The Casimir operators are

$$\begin{aligned} p^2 &= p^\mu p_\mu \\ W^2 &= W_\mu W^\mu \end{aligned} \quad (2.3)$$

where $W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} p_\nu M_{\rho\sigma}$. These operators commute not only with the members of the Poincaré group, but also with the internal symmetry generators,

$$[B_r, p^2] = [B_r, W^2] = 0 \quad (2.4)$$

This implies two things: First, all members of the same irreducible multiplet of the internal group have the same mass [21] and secondly that they are bound to have the same spin. In four-dimensional Weyl spinor notation, the $\mathcal{N}$ supersymmetry generators are denoted by $Q_{\alpha i}$, $i = 1 \dots \mathcal{N}$. The fermionic supersymmetry generators satisfy

$$\begin{aligned} [Q_{\alpha i}, M_{\mu\nu}] &= \frac{1}{2} (\sigma_{\mu\nu})_\alpha^\beta Q_{\beta i} \\ [\bar{Q}_\alpha^i, M_{\mu\nu}] &= \frac{1}{2} \bar{Q}_\beta^i (\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \end{aligned} \quad (2.5)$$

We also have the key anticommutator,

$$\{Q_\alpha, \bar{Q}^\beta\} = (\gamma^\mu)_\alpha^\beta p_\mu \quad (2.6)$$

with the γ^μ defined by

$$(\gamma^\mu)_{\alpha\beta} = -(\gamma^\mu)_{\beta\alpha} , \quad (2.7)$$

and

$$Tr(\gamma^\mu \gamma^\nu) = 4\eta^{\mu\nu} . \quad (2.8)$$

2.2 Supersymmetry and the Light-Cone

We continue working with the metric $(-, +, +, \dots, +)$. As a first step to formulating our theory in light-cone gauge, we introduce the following light-cone coordinates and their derivatives.

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^3) ; \quad \partial^\pm = \frac{1}{\sqrt{2}}(-\partial_0 \pm \partial_3) ; \quad (2.9)$$

$$x = \frac{1}{\sqrt{2}}(x_1 + i x_2) ; \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_1 - i \partial_2) ; \quad (2.10)$$

$$\bar{x} = \frac{1}{\sqrt{2}}(x_1 - i x_2) ; \quad \partial = \frac{1}{\sqrt{2}}(\partial_1 + i \partial_2) , \quad (2.11)$$

such that

$$\partial^+ x^- = \partial^- x^+ = -1 ; \quad \bar{\partial} x = \partial \bar{x} = +1 . \quad (2.12)$$

This then yields the following 4-product,

$$A^\mu B_\mu = A_i B_i - A_+ B_- - A_- B_+ \quad (2.13)$$

In the light-cone frame, $p^- = -i \frac{\partial}{\partial x^+}$ is interpreted as the Hamiltonian. We will consider the generators of the SuperPoincaré algebra at the light-cone time $x^+ = 0$. Generators that are capable of shoftong away from a given x^+ surface are referred to as dynamical. The rest are kinematical. We use $m, n, p, q \dots$ vector indices to denote

transverse space-time indices. These run over $SO(d-2)$ values in d -dimensions. In order to be able to translate between spinor and vector notation, we define gamma-matrices by

$$(\gamma^m)^{\alpha\beta} = -(\gamma^m)^{\beta\alpha} , \quad (2.14)$$

such that

$$Tr(\gamma^m \gamma^n) = (d-2) \delta^{mn}. \quad (2.15)$$

This then allows us to write,

$$C^{\alpha\beta} \equiv C_m (\gamma^m)^{\alpha\beta} . \quad (2.16)$$

We also define

$$\gamma^{mn} = \frac{1}{2} [\gamma^m, \gamma^n], \quad m \neq n. \quad (2.17)$$

We now project out equation (2.6), using $\frac{1}{2} \gamma_+ \gamma_-$ to obtain

$$\{Q_{+\alpha}, \bar{Q}_+^{\beta}\} = (\gamma_+)^{\beta}_{\alpha} p^+ \quad (2.18)$$

which represents the “restricted” supersymmetry algebra. We simply pick a specific representation for the γ -matrices to read-off the light-cone supersymmetry algebra,

$$\begin{aligned} \{q_+^{\alpha}, \bar{q}_{+\beta}\} &= -\sqrt{2} \delta_{\beta}^{\alpha} p^+ , \\ \{q_-^{\alpha}, \bar{q}_{-\beta}\} &= -\sqrt{2} \delta_{\beta}^{\alpha} p^- , \\ \{q_+^{\alpha}, \bar{q}_{-\beta}\} &= -\sqrt{2} \delta_{\beta}^{\alpha} p . \end{aligned} \quad (2.19)$$

The q_+ part of the algebra is linearly realized in superspace, characterized by a Grassmann parameter θ^α and its complex conjugate $\bar{\theta}_\alpha$. These variables are defined by

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_\alpha, \bar{\theta}_\beta\} = \{\bar{\theta}_\alpha, \theta^\beta\} = 0, \quad (2.20)$$

The corresponding Grassmann derivatives are written as

$$\bar{\partial}_\alpha \equiv \frac{\partial}{\partial \theta^\alpha}; \quad \partial^\beta \equiv \frac{\partial}{\partial \bar{\theta}_\beta}, \quad (2.21)$$

with canonical anticommutation relations

$$\{\partial^\alpha, \bar{\theta}_\beta\} = \delta^\alpha_\beta; \quad \{\bar{\partial}_\alpha, \theta^\beta\} = \delta_\alpha^\beta. \quad (2.22)$$

Under conjugation, upper and lower spinor indices are interchanged, so that $\overline{\theta^\alpha} = \bar{\theta}_\alpha$, while

$$\overline{(\bar{\partial}_\alpha)} = -\partial^\alpha; \quad \overline{(\partial^\beta)} = -\bar{\partial}_\beta. \quad (2.23)$$

Also, the order of the operators is interchanged; that is $\overline{\theta^\alpha \theta^\beta} = \bar{\theta}_\beta \bar{\theta}_\alpha$ and $\overline{\partial^\alpha \partial^\beta} = \bar{\partial}_\beta \bar{\partial}_\alpha$.

The kinematical supersymmetry is then realized by choosing

$$q_+^\alpha = -\partial^\alpha + \frac{i}{\sqrt{2}} \theta^\alpha \partial^+; \quad \bar{q}_{+\beta} = \bar{\partial}_\beta - \frac{i}{\sqrt{2}} \bar{\theta}_\beta \partial^+, \quad (2.24)$$

so that they satisfy

$$\{q_+^\alpha, \bar{q}_{+\beta}\} = i\sqrt{2} \delta^\alpha_\beta \partial^+, \quad (2.25)$$

The dynamical supersymmetries are obtained in the next section (these are obtained using the light-cone superspace Poincaré algebra which we need to define first). These

fermionic variables are also used to define chiral derivatives in superspace,

$$d^\alpha = -\partial^\alpha - \frac{i}{\sqrt{2}} \theta^\alpha \partial^+; \quad \bar{d}_\beta = \bar{\partial}_\beta + \frac{i}{\sqrt{2}} \bar{\theta}_\beta \partial^+, \quad (2.26)$$

which satisfy

$$\{d^\alpha, \bar{d}_\beta\} = -i\sqrt{2}\delta^\alpha_\beta \partial^+. \quad (2.27)$$

2.3 The Complete $d = 4$ SuperPoincaré Algebra

The entire (superspace)light-cone SuperPoincaré algebra, splits up into kinematical and dynamical pieces. Apart from a few minor alterations, this algebra is presented in reference [12]. The kinematical generators are

- the three momenta,

$$p^+ = -i\partial^+, \quad p = -i\partial, \quad \bar{p} = -i\bar{\partial}, \quad (2.28)$$

- the transverse space rotation,

$$j = x\bar{\partial} - \bar{x}\partial + S^{12}, \quad (2.29)$$

where

$$S^{12} = \frac{1}{2}(\theta^\alpha \bar{\partial}_\alpha - \bar{\theta}_\alpha \partial^\alpha) - \frac{i}{4\sqrt{2}\partial^+}(d^\alpha \bar{d}_\alpha - \bar{d}_\alpha d^\alpha). \quad (2.30)$$

The last term is included to ensure that these generators commute with the chiral derivatives (this is related to chirality-preservation, when acting on superfields. We will discuss this when we introduce the superfields),

$$[j, d^\alpha] = [j, \bar{d}_\beta] = 0. \quad (2.31)$$

- and the “plus-rotations”

$$j^+ = i x \partial^+ , \quad \bar{j}^+ = i \bar{x} \partial^+ . \quad (2.32)$$

$$j^{+-} = i x^- \partial^+ - \frac{i}{2} (\theta^\alpha \bar{\partial}_\alpha + \bar{\theta}_\alpha \partial^\alpha) , \quad (2.33)$$

which satisfy

$$[j^{+-}, d^\alpha] = \frac{i}{2} d^\alpha , \quad [j^{+-}, \bar{d}_\beta] = \frac{i}{2} \bar{d}_\beta , \quad (2.34)$$

and thus preserve the chiral derivatives. Note that it is only for the choice $x^+ = 0$ that the generator j^{+-} is kinematical, since the dynamical part is multiplied by $x^+ = 0$.

The dynamical generators are

- the light-cone Hamiltonian,

$$p^- = -i \frac{\partial}{\partial} \partial^+ \quad (2.35)$$

- and the dynamical boosts,

$$\begin{aligned} j^- &= i x \frac{\partial \bar{\partial}}{\partial^+} - i x^- \partial + i \left(\theta^\alpha \bar{\partial}_\alpha + \frac{i}{4\sqrt{2} \partial^+} (d^\alpha \bar{d}_\alpha - \bar{d}_\alpha d^\alpha) \right) \frac{\partial}{\partial^+} , \\ \bar{j}^- &= i \bar{x} \frac{\partial \bar{\partial}}{\partial^+} - i x^- \bar{\partial} + i \left(\bar{\theta}_\beta \partial^\beta + \frac{i}{4\sqrt{2} \partial^+} (d^\beta \bar{d}_\beta - \bar{d}_\beta d^\beta) \right) \frac{\bar{\partial}}{\partial^+} . \end{aligned} \quad (2.36)$$

The helicity counter added to the expressions above implies that

$$[j^-, d^\alpha] = \frac{i}{2} d^\alpha \frac{\partial}{\partial^+} , \quad [j^-, \bar{d}_\beta] = \frac{i}{2} \bar{d}_\beta \frac{\partial}{\partial^+} . \quad (2.37)$$

These generators also satisfy

$$[j^-, \bar{j}^+] = -ij^{+-} - j, \quad [j^-, j^{+-}] = ij^- . \quad (2.38)$$

In a similar fashion, the supersymmetries split into

- the already introduced kinematical supersymmetries,

$$q_+^\alpha = -\partial^\alpha + \frac{i}{\sqrt{2}} \theta^\alpha \partial^+ ; \quad \bar{q}_{+\beta} = \bar{\partial}_\beta - \frac{i}{\sqrt{2}} \bar{\theta}_\beta \partial^+ , \quad (2.39)$$

satisfying

$$\{q_+^\alpha, \bar{q}_{+\beta}\} = i\sqrt{2}\delta^\alpha_\beta \partial^+ , \quad (2.40)$$

and anticommuting with the chiral derivatives

$$\{q_+^\alpha, \bar{d}_\beta\} = \{d^\alpha, \bar{q}_{+\beta}\} = 0 . \quad (2.41)$$

- dynamical supersymmetries, which may be obtained by boosting the kinematical ones

$$q_-^\alpha \equiv i[\bar{j}^-, q_+^\alpha] = \frac{\bar{\partial}}{\partial^+} q_+^\alpha , \quad \bar{q}_{-\beta} \equiv i[j^-, \bar{q}_{+\beta}] = \frac{\partial}{\partial^+} \bar{q}_{+\beta} . \quad (2.42)$$

They obey

$$\{q_-^\alpha, \bar{q}_{-\beta}\} = i\sqrt{2}\delta^\alpha_\beta \frac{\partial}{\bar{\partial}} \partial^+ , \quad (2.43)$$

and

$$\{q_+^\alpha, \bar{q}_{-\beta}\} = i\sqrt{2}\delta^\alpha_\beta \partial . \quad (2.44)$$

CHAPTER 3 THE YANG-MILLS SYSTEM

Having set up the relevant tools in superspace we now focus on formulating our theories in that space. Both the $(\mathcal{N} = 1, d = 10)$ and $(\mathcal{N} = 4, d = 4)$ SuperYang-Mills theories are known in component form (in the light-cone gauge). However, while the four-dimensional version also has a superspace description, the $d = 10$ variety does not. This is the problem we resolve in this section. After a brief review of the four-dimensional theory (in superspace), we oxidize it up to ten dimensions thus obtaining a complete light-cone superspace description of $(\mathcal{N} = 1, d = 10)$ Yang-Mills.

3.1 Ten-Dimensional Supersymmetric Yang-Mills

A Yang-Mills theory based on a local symmetry G is a field theory with symmetry currents coupled minimally to vector-boson fields. The space integrals of the time components of the currents formally define the generators T^a of the symmetry group G . The generators of the group and consequently the currents from which they are constructed transform as the adjoint representation. For the current vector boson coupling to be invariant under G the bosons must also transform as the adjoint.

Leptons and quarks which are spin $\frac{1}{2}$ fermions are usually assumed to be fundamental fields in the Lagrangian. The generators T^a of the Lie group G satisfy

$$[T^a, T^b] = i f^{abc} T^c \tag{3.1}$$

where the f^{abc} are structure constants. Mathematically, gauge fields are introduced through the covariant derivative

$$D_\mu = \partial_\mu + ig A_\mu^a T^a \tag{3.2}$$

from which we construct the field strengths

$$F_{\mu\nu} = F_{\mu\nu}{}^a T^a = -i[D_\mu, D_\nu] \quad (3.3)$$

The $d = 10$ Yang Mills action reads

$$\mathcal{S} = \int d^{10}x \left\{ -\frac{1}{4} F_{\mu\nu}{}^a F^{\mu\nu}{}_a + \frac{i}{2} \bar{\lambda}^a \gamma^\mu D_\mu \lambda^a \right\} \quad (3.4)$$

where the vector and spinor fields transform according to the adjoint representation of some semisimple Lie group. The spinor satisfies both the Majorana and Weyl constraints.

This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta A_\mu{}^a &= i \bar{\alpha} \gamma_\mu \lambda^a \\ \delta \lambda^a &= -\frac{1}{2} \gamma^{\mu\nu} \alpha F_{\mu\nu}{}^a \end{aligned} \quad (3.5)$$

Light cone gauge is fixed by choosing $A_-{}^a = 0$. The spinor breaks up as

$$\lambda = \frac{1}{2} (\gamma_+ \gamma_- + \gamma_- \gamma_+) \lambda = \lambda_+ + \lambda_- \quad (3.6)$$

As an illustration of the LC_2 approach, we focus purely on the bosonic portion of the above action. The bosonic equations of motion corresponding are

$$\partial_\nu F^{\mu\nu a} + g f^{abc} F^{\mu\nu b} A_\nu{}^c = 0 \quad (3.7)$$

LC_2 indicates that all spurious degrees of freedom have been eliminated using the equations of motion. For example, the $\mu = +$ component of this equation implies

that

$$A_+{}^a = \frac{1}{\partial_-}(\partial_i A_i{}^a) + g f^{abc} \frac{1}{\partial_-^2} \{(\partial_- A_k{}^b) A_k{}^c\} \quad (3.8)$$

When substituted back into the bosonic part of the action we obtain

$$\begin{aligned} \mathcal{S} = \int d^{10}x \{ & \frac{1}{2} A_i{}^a \square A_i{}^a \\ & + g f^{abc} [\partial_i A_i{}^a \frac{1}{\partial^+} (\partial^+ A_j{}^b A_j{}^c) - \partial_i A_j{}^a A_i{}^b A_j{}^c] \\ & + g^2 f^{abc} f^{ade} [-\frac{1}{4} A_i{}^b A_j{}^c A_i{}^d A_j{}^e - \frac{1}{2} \frac{1}{\partial^+} (\partial^+ A_i{}^b A_i{}^c) \frac{1}{\partial^+} (\partial^+ A_j{}^d A_j{}^e)] \} \end{aligned} \quad (3.9)$$

This procedure is easily extended to include the fermions. The $\frac{1}{\partial^+}$ may be thought of as a Green's function that satisfies

$$\partial^+ G(x^-) = \delta(x^-)$$

This determines G upto a factor h where $\partial^+ h = 0$. i.e. up to a zero mode $h = h(x^+)$ of the operator ∂^+ (for more details on this prescription, we refer the reader to reference [22]).

3.2 ($\mathcal{N} = 4, d = 4$) SuperYang-Mills in Light-Cone Superspace

When ten-dimensional $\mathcal{N} = 1$ Yang-Mills is dimensionally reduced to four dimensions it yields one complex bosonic field (the gauge field), four complex Grassmann fields and six scalars. In four dimensions any massless particle can be described by a complex field and its complex conjugate of opposite helicity, the $SO(2)$ coming from the little group decomposition

$$SO(8) \supset SO(2) \times SO(6) . \quad (3.10)$$

Particles with no helicity are described by real fields. The eight vectors fields in ten dimension reduce to

$$\mathbf{8}_v = \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1} , \quad (3.11)$$

and the eight spinors to

$$\mathbf{8}_s = \mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2} . \quad (3.12)$$

The representations on the right-hand side belong to $SO(6) \sim SU(4)$ with subscripts denoting the helicity: there are six scalar fields, two vector fields, four spinor fields and their conjugates. We introduce the anticommuting Grassmann variables θ^α and $\bar{\theta}_\beta$

$$\{\theta^\alpha, \theta^\beta\} = \{\bar{\theta}_\alpha, \bar{\theta}_\beta\} = \{\bar{\theta}_\alpha, \theta^\beta\} = 0 , \quad (3.13)$$

which transform as the spinor representations of $SO(6) \sim SU(4)$

$$\theta^\alpha \sim \mathbf{4}_{1/2} ; \quad \bar{\theta}_\beta \sim \bar{\mathbf{4}}_{-1/2} , \quad (3.14)$$

where $\alpha, \beta, \gamma, \delta, \dots = 1, 2, 3, 4$, denote $SU(4)$ spinor indices. Their derivatives are written as

$$\bar{\partial}_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} ; \quad \partial^\beta \equiv \frac{\partial}{\partial \bar{\theta}_\beta} , \quad (3.15)$$

with canonical anticommutation relations

$$\{\partial^\alpha, \bar{\theta}_\beta\} = \delta^\alpha_\beta ; \quad \{\bar{\partial}_\alpha, \theta^\beta\} = \delta_\alpha^\beta . \quad (3.16)$$

Having introduced these variables, we may now capture it all the physical degrees of freedom of the ($\mathcal{N} = 4, d = 4$) theory in one complex superfield [12]

$$\begin{aligned} \phi(y) = & \frac{1}{\partial^+} A(y) + \frac{i}{\sqrt{2}} \theta^\alpha \theta^\beta \bar{C}_{\alpha\beta}(y) + \frac{1}{12} \theta^\alpha \theta^\beta \theta^\gamma \theta^\delta \epsilon_{\alpha\beta\gamma\delta} \partial^+ \bar{A}(y) \\ & + \frac{i}{\partial^+} \theta^\alpha \bar{\chi}_\alpha(y) + \frac{\sqrt{2}}{6} \theta^\alpha \theta^\beta \theta^\gamma \epsilon_{\alpha\beta\gamma\delta} \chi^\delta(y) . \end{aligned} \quad (3.17)$$

In this notation, the eight original gauge fields $A_i, i = 1, \dots, 8$ appear as

$$A = \frac{1}{\sqrt{2}}(A_1 + i A_2), \quad \bar{A} = \frac{1}{\sqrt{2}}(A_1 - i A_2), \quad (3.18)$$

while the six scalar fields are written as antisymmetric $SU(4)$ bi-spinors

$$C^{\alpha 4} = \frac{1}{\sqrt{2}}(A_{\alpha+3} + i A_{\alpha+6}), \quad \bar{C}^{\alpha 4} = \frac{1}{\sqrt{2}}(A_{\alpha+3} - i A_{\alpha+6}), \quad (3.19)$$

for $\alpha \neq 4$; complex conjugation is akin to duality

$$\bar{C}_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} C^{\gamma\delta}. \quad (3.20)$$

The fermion fields are denoted by χ^α and $\bar{\chi}_\alpha$. All have adjoint indices (not shown here) and are local fields in the modified light-cone coordinates

$$y = (x, \bar{x}, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^\alpha \bar{\theta}_\alpha). \quad (3.21)$$

In this LC_2 light-cone formulation all the unphysical degrees of freedom have been integrated out leaving only the physical ones.

One verifies that ϕ and its complex conjugate $\bar{\phi}$ satisfy the chiral constraints

$$d^\alpha \phi = 0; \quad \bar{d}_\alpha \bar{\phi} = 0, \quad (3.22)$$

as well as the “inside-out” constraints

$$\bar{d}_\alpha \bar{d}_\beta \phi = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} d^\gamma d^\delta \bar{\phi}, \quad (3.23)$$

$$d^\alpha d^\beta \bar{\phi} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \bar{d}_\gamma \bar{d}_\delta \phi. \quad (3.24)$$

The ($\mathcal{N} = 4, d = 4$) Yang-Mills action is then simply

$$\int d^4x \int d^4\theta d^4\bar{\theta} \mathcal{L} , \quad (3.25)$$

where

$$\begin{aligned} \mathcal{L} = & -\bar{\phi} \frac{\square}{\partial^{+2}} \phi + \frac{4g}{3} f^{abc} \left(\frac{1}{\partial^{+}} \bar{\phi}^a \phi^b \bar{\partial} \phi^c + \frac{1}{\partial^{+}} \phi^a \bar{\phi}^b \partial \bar{\phi}^c \right) \\ & -g^2 f^{abc} f^{ade} \left(\frac{1}{\partial^{+}} (\phi^b \partial^{+} \phi^c) \frac{1}{\partial^{+}} (\bar{\phi}^d \partial^{+} \bar{\phi}^e) + \frac{1}{2} \phi^b \bar{\phi}^c \phi^d \bar{\phi}^e \right) . \end{aligned} \quad (3.26)$$

Grassmann integration is normalized so that $\int d^4\theta \theta^1 \theta^2 \theta^3 \theta^4 = 1$, and f^{abc} are the structure functions of the Lie algebra.

3.3 Oxidizing ($\mathcal{N} = 4, d = 4$) SuperYang-Mills to ($\mathcal{N} = 1, d = 10$) Yang-Mills

Apart from some minor changes most of the material presented up to this point appears in references [11] and [12]. We now embark on the program outlined in our overview: we will show that the compact $d = 4$ formalism discussed so far can be generalized to entirely describe the $d = 10$ theory.

3.3.1 Ten Dimensions

The very compact formalism of the previous section is specific to the $N = 4$ theory in four dimensions. We now generalize this formalism to restore the theory to ten dimensions without altering the superfield (except for an added dependence on new coordinates). This is achieved simply by introducing generalized derivative operators.

First of, the transverse light-cone variables need to be generalized to eight. We stick to the previous notation and introduce the six extra coordinates and their

derivatives as antisymmetric bi-spinors

$$x^{\alpha 4} = \frac{1}{\sqrt{2}} (x_{\alpha+3} + i x_{\alpha+6}) , \quad \partial^{\alpha 4} = \frac{1}{\sqrt{2}} (\partial_{\alpha+3} + i \partial_{\alpha+6}) , \quad (3.27)$$

for $\alpha \neq 4$, and their complex conjugates

$$\bar{x}_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} x^{\alpha\beta} ; \quad \bar{\partial}_{\gamma\delta} = \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} \partial^{\alpha\beta} . \quad (3.28)$$

Their derivatives satisfy

$$\bar{\partial}_{\alpha\beta} x^{\gamma\delta} = (\delta_{\alpha}^{\gamma} \delta_{\beta}^{\delta} - \delta_{\alpha}^{\delta} \delta_{\beta}^{\gamma}) ; \quad \partial^{\alpha\beta} \bar{x}_{\gamma\delta} = (\delta^{\alpha}_{\gamma} \delta^{\beta}_{\delta} - \delta^{\alpha}_{\delta} \delta^{\beta}_{\gamma}) , \quad (3.29)$$

and

$$\partial^{\alpha\beta} x^{\gamma\delta} = \frac{1}{2} \epsilon^{\gamma\delta\rho\sigma} \partial^{\alpha\beta} \bar{x}_{\rho\sigma} = \epsilon^{\alpha\beta\rho\sigma} . \quad (3.30)$$

There are no modifications to be made to the chiral superfield except for the dependence on the extra coordinates

$$A(y) = A(x, \bar{x}, x^{\alpha\beta}, \bar{x}_{\alpha\beta}, y^{-}) , \quad etc... . \quad (3.31)$$

These extra variables will be acted on by new operators that generate the higher-dimensional symmetries.

3.3.2 The SuperPoincaré Algebra in 10 Dimensions

We start with the construction of the $SO(8)$ little group using the decomposition $SO(8) \supset SO(2) \times SO(6)$. The $SO(2)$ generator is the same as before while the $SO(6) \sim SU(4)$ generators are given by

$$J^{\alpha}_{\beta} = \frac{1}{2} (x^{\alpha\rho} \bar{\partial}_{\rho\beta} - \bar{x}_{\rho\beta} \partial^{\alpha\rho}) - \theta^{\alpha} \bar{\partial}_{\beta} + \bar{\theta}_{\beta} \partial^{\alpha} + \frac{1}{4} (\theta^{\rho} \bar{\partial}_{\rho} - \bar{\theta}_{\rho} \partial^{\rho}) \delta^{\alpha}_{\beta}$$

$$+\frac{i}{2\sqrt{2}\partial^+}(d^\alpha\bar{d}_\beta-\bar{d}_\beta d^\alpha)+\frac{i}{8\sqrt{2}\partial^+}(d^\rho\bar{d}_\rho-\bar{d}_\rho d^\rho)\delta^\alpha{}_\beta. \quad (3.32)$$

The extra terms with the d and $\bar{d}$ operators are not necessary for closure of the algebra however they ensure that the generators commute with the chiral derivatives. They satisfy the commutation relations

$$\left[J, J^\alpha{}_\beta\right] = 0, \quad \left[J^\alpha{}_\beta, J^\rho{}_\sigma\right] = \delta^\alpha{}_\sigma J^\rho{}_\beta - \delta^\rho{}_\beta J^\alpha{}_\sigma. \quad (3.33)$$

The remaining $SO(8)$ generators lie in the coset $SO(8)/(SO(2) \times SO(6))$

$$\begin{aligned} J^{\rho\sigma} &= x\partial^{\rho\sigma} - x^{\rho\sigma}\partial + \frac{i}{\sqrt{2}}\partial^+\theta^\rho\theta^\sigma - i\sqrt{2}\frac{1}{\partial^+}\partial^\rho\partial^\sigma + \frac{i}{\sqrt{2}\partial^+}d^\rho d^\sigma, \\ \bar{J}_{\alpha\beta} &= \bar{x}\bar{\partial}_{\alpha\beta} - \bar{x}_{\alpha\beta}\bar{\partial} + \frac{i}{\sqrt{2}}\partial^+\bar{\theta}_\alpha\bar{\theta}_\beta - i\sqrt{2}\frac{1}{\partial^+}\bar{\partial}_\alpha\bar{\partial}_\beta + \frac{i}{\sqrt{2}\partial^+}\bar{d}_\alpha\bar{d}_\beta. \end{aligned} \quad (3.34)$$

All $SO(8)$ transformations are specially constructed so as not to mix chiral and antichiral superfields

$$[J^{\alpha\beta}, \bar{d}_\rho] = 0; \quad [\bar{J}_{\alpha\beta}, d^\rho] = 0, \quad (3.35)$$

and satisfy the $SO(8)$ commutation relations

$$\begin{aligned} \left[J, J^{\alpha\beta}\right] &= J^{\alpha\beta}, \quad \left[J, \bar{J}_{\alpha\beta}\right] = -\bar{J}_{\alpha\beta}, \\ \left[J^\alpha{}_\beta, J^{\rho\sigma}\right] &= \delta^\sigma{}_\beta J^{\alpha\rho} - \delta^\rho{}_\beta J^{\alpha\sigma}, \quad \left[J^\alpha{}_\beta, \bar{J}_{\rho\sigma}\right] = \delta^\alpha{}_\sigma \bar{J}_{\beta\rho} - \delta^\alpha{}_\rho \bar{J}_{\beta\sigma}, \\ \left[J^{\alpha\beta}, \bar{J}_{\rho\sigma}\right] &= \delta^\alpha{}_\rho J^\beta{}_\sigma + \delta^\beta{}_\sigma J^\alpha{}_\rho - \delta^\beta{}_\rho J^\alpha{}_\sigma - \delta^\alpha{}_\sigma J^\beta{}_\rho - (\delta^\alpha{}_\rho \delta^\beta{}_\sigma - \delta^\beta{}_\rho \delta^\alpha{}_\sigma)J. \end{aligned}$$

Rotations between the 1 or 2 and 4 through 9 directions induce on the chiral fields the changes

$$\delta \phi = \left(\frac{1}{2} \omega_{\alpha\beta} J^{\alpha\beta} + \frac{1}{2} \bar{\omega}^{\alpha\beta} \bar{J}_{\alpha\beta} \right) \phi, \quad (3.36)$$

where complex conjugation is like duality

$$\bar{\omega}_{\rho\sigma} = \frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} \omega^{\alpha\beta}. \quad (3.37)$$

For example, a rotation in the 1 – 4 plane through an angle θ corresponds to taking $\theta = \omega_{14} = \omega_{23}$ ($= \omega^{23} = \omega^{14}$ by reality), all other components being zero. Finally, we verify that the kinematical supersymmetries are duly rotated by these generators

$$[J^{\alpha\beta}, \bar{q}_{+\rho}] = \delta^\beta_\rho q_+^\alpha - \delta^\alpha_\rho q_+^\beta; \quad [\bar{J}_{\alpha\beta}, q_+^\rho] = \delta_{\beta}^\rho \bar{q}_{+\alpha} - \delta_{\alpha}^\rho \bar{q}_{+\beta}. \quad (3.38)$$

We now use the $SO(8)$ generators to construct the other SuperPoincaré generators

$$J^+ = i x \partial^+; \quad \bar{J}^+ = i \bar{x} \partial^+$$

$$J^{+\alpha\beta} = i x^{\alpha\beta} \partial^+; \quad \bar{J}^+_{\alpha\beta} = i \bar{x}_{\alpha\beta} \partial^+. \quad (3.39)$$

The dynamical boosts are now

$$\begin{aligned} J^- = i x \frac{\partial \bar{\partial} + \frac{1}{4} \bar{\partial}_{\rho\sigma} \partial^{\rho\sigma}}{\partial^+} - i x^- \partial + i \frac{\partial}{\partial^+} \left\{ \theta^\alpha \bar{\partial}_\alpha + \frac{i}{4\sqrt{2} \partial^+} (d^\rho \bar{d}_\rho - \bar{d}_\rho d^\rho) \right\} - \\ - \frac{1}{4} \frac{\bar{\partial}_{\rho\sigma}}{\partial^+} \left\{ \frac{\partial^+}{\sqrt{2}} \theta^\rho \theta^\sigma - \frac{\sqrt{2}}{\partial^+} \partial^\rho \partial^\sigma + \frac{1}{\sqrt{2} \partial^+} d^\rho d^\sigma \right\}, \end{aligned} \quad (3.40)$$

and its conjugate

$$\begin{aligned} \bar{J}^- = & i \bar{x} \frac{\partial \bar{\partial} + \frac{1}{4} \bar{\partial}_{\rho\sigma} \partial^{\rho\sigma}}{\partial^+} - i x^- \bar{\partial} + i \frac{\bar{\partial}}{\partial^+} \left\{ \bar{\theta}_\alpha \partial^\alpha + \frac{i}{4\sqrt{2}\partial^+} (d^\rho \bar{d}_\rho - \bar{d}_\rho d^\rho) \right\} - \\ & - \frac{1}{4} \frac{\partial^{\rho\sigma}}{\partial^+} \left\{ \frac{\partial^+}{\sqrt{2}} \bar{\theta}_\rho \bar{\theta}_\sigma - \frac{\sqrt{2}}{\partial^+} \bar{\partial}_\rho \bar{\partial}_\sigma + \frac{1}{\sqrt{2}\partial^+} \bar{d}_\rho \bar{d}_\sigma \right\}. \end{aligned} \quad (3.41)$$

The others are obtained by using the $SO(8)/(SO(2) \times SO(6))$ rotations

$$J^{-\alpha\beta} = [J^-, J^{\alpha\beta}]; \quad \bar{J}_{\alpha\beta} = [\bar{J}^-, \bar{J}_{\alpha\beta}]. \quad (3.42)$$

We do not show their explicit forms as they are too cumbersome. The four supersymmetries in four dimensions turn into one supersymmetry in ten dimensions. In our notation, the kinematical supersymmetries q_+^α and $\bar{q}_{+\alpha}$, are assembled into one $SO(8)$ spinor. The dynamical supersymmetries are obtained by boosting

$$i[\bar{J}^-, q_+^\alpha] \equiv \mathcal{Q}^\alpha, \quad i[J^-, \bar{q}_{+\alpha}] \equiv \bar{\mathcal{Q}}_\alpha, \quad (3.43)$$

where

$$\begin{aligned} \mathcal{Q}^\alpha &= \frac{\bar{\partial}}{\partial^+} q_+^\alpha + \frac{1}{2} \frac{\partial^{\alpha\beta}}{\partial^+} \bar{q}_{+\beta}, \\ \bar{\mathcal{Q}}_\alpha &= \frac{\partial}{\partial^+} \bar{q}_{+\alpha} + \frac{1}{2} \frac{\bar{\partial}_{\alpha\beta}}{\partial^+} q_+^\beta. \end{aligned} \quad (3.44)$$

They satisfy the supersymmetry algebra

$$\{\mathcal{Q}^\alpha, \bar{\mathcal{Q}}_\beta\} = i\sqrt{2}\delta_\beta^\alpha \frac{1}{\partial^+} \left(\partial \bar{\partial} + \frac{1}{4} \bar{\partial}_{\rho\sigma} \partial^{\rho\sigma} \right), \quad (3.45)$$

and can be obtained from one another by $SO(8)$ rotations

$$\frac{1}{2} \epsilon_{\alpha\beta\rho\sigma} [J^{\rho\sigma}, \mathcal{Q}^\alpha] = 4 \overline{\mathcal{Q}}_\beta, \quad (3.46)$$

while

$$[\bar{J}_{\rho\sigma}, \mathcal{Q}^\alpha] = 0. \quad (3.47)$$

Note also that

$$\{\mathcal{Q}^\alpha, q_+^\beta\} = \frac{i}{\sqrt{2}} \partial^{\alpha\beta}. \quad (3.48)$$

This framework offers a simple method to introduce the central charges germane to the four-dimensional theory: the six-dimensional derivatives $\partial^{\alpha\beta}$ are simply replaced by c-numbers $Z^{\alpha\beta}$ thus yielding the massive supersymmetry algebra in four dimensions with six central charges

$$Z^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \overline{Z}_{\rho\sigma}.$$

However, relaxing this duality condition gives us all 12 central charges of the $N = 4$ Yang-Mills Theory thus indicating that both varieties of central charges have a common origin in the light-cone gauge formulation.

Starting with these tools we proceed to build the interacting theory in ten dimensions.

3.3.3 The Generalized Derivatives

The cubic interaction in the ($\mathcal{N} = 4, d = 4$) Lagrangian contains explicitly the derivative operators ∂ and $\bar{\partial}$. To achieve covariance in ten dimensions these must be generalized. We propose the following operator

$$\overline{\nabla} \equiv \bar{\partial} + \frac{i\alpha}{4\sqrt{2}\partial^+} \bar{d}_\rho \bar{d}_\sigma \partial^{\rho\sigma}, \quad (3.49)$$

which naturally incorporates the rest of the derivatives $\partial^{\rho\sigma}$ with α as an arbitrary parameter. After some algebra we find that $\bar{\nabla}$ is covariant under $SO(8)$ transformations. We define its rotated partner by

$$\nabla^{\alpha\beta} \equiv \left[\bar{\nabla}, J^{\alpha\beta} \right], \quad (3.50)$$

where

$$\nabla^{\alpha\beta} = \partial^{\alpha\beta} - \frac{i\alpha}{4\sqrt{2}\partial^+} \bar{d}_\rho \bar{d}_\sigma \epsilon^{\alpha\beta\rho\sigma} \partial. \quad (3.51)$$

If we apply to it the inverse transformation, it goes back to the original form

$$\left[\bar{J}_{\rho\sigma}, \nabla^{\alpha\beta} \right] = (\delta_\rho^\alpha \delta_\sigma^\beta - \delta_\sigma^\alpha \delta_\rho^\beta) \bar{\nabla}, \quad (3.52)$$

and these operators transform under $SO(8)/(SO(2) \times SO(6))$, and $SO(2) \times SO(6)$ as the components of an 8-vector.

We introduce the conjugate operator ∇ by requiring that

$$\nabla \bar{\phi} \equiv \overline{(\bar{\nabla} \phi)}, \quad (3.53)$$

with

$$\nabla \equiv \partial + \frac{i\alpha}{4\sqrt{2}\partial^+} d^\rho d^\sigma \bar{\partial}_{\rho\sigma}. \quad (3.54)$$

Define

$$\bar{\nabla}_{\alpha\beta} \equiv \left[\nabla, \bar{J}_{\alpha\beta} \right], \quad (3.55)$$

which is given by

$$\bar{\nabla}_{\alpha\beta} = \bar{\partial}_{\alpha\beta} - \frac{i\alpha}{4\sqrt{2}\partial^+} d^\rho d^\sigma \epsilon_{\alpha\beta\rho\sigma} \bar{\partial}. \quad (3.56)$$

We then verify that

$$\left[J^{\alpha\beta}, \bar{\nabla}_{\rho\sigma} \right] = (\delta_\rho^\alpha \delta_\sigma^\beta - \delta_q^\alpha \delta_\rho^\beta) \nabla. \quad (3.57)$$

The value of the parameter α is fixed by requiring invariance of the cubic interaction.

3.3.4 Invariance of the Action

The kinetic term is trivially made $SO(8)$ -invariant by including the six extra transverse derivatives in the d'Alembertian. The quartic interactions are obviously invariant since they do not contain any transverse derivative operators. Hence we need only consider the cubic vertex.

The whole point of this exercise is to show that covariance in ten dimensions is achieved by simply replacing the transverse derivatives ∂ and $\bar{\partial}$ by ∇ and $\bar{\nabla}$ respectively. We thus propose the new cubic interaction term

$$\frac{4g}{3} f^{abc} \int d^{10}x \int d^4\theta d^4\bar{\theta} \left(\frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\nabla} \phi^c + \frac{1}{\partial^+} \phi^a \bar{\phi}^b \nabla \bar{\phi}^c \right) \quad (3.58)$$

Since it is obviously invariant under $SO(6) \times SO(2)$ we need only consider the coset variations. Acting on the chiral superfield we have

$$\delta_J \phi \equiv \bar{\omega}_{\alpha\beta} J^{\alpha\beta} \phi = i\sqrt{2} \bar{\omega}_{\alpha\beta} \partial^+ \theta^\alpha \theta^\beta \phi, \quad (3.59)$$

$$\delta_J \phi \equiv \omega^{pq} \bar{J}_{pq} \phi = \omega^{pq} \left\{ \frac{i}{\sqrt{2}} \partial^+ \bar{\theta}_p \bar{\theta}_q - i\sqrt{2} \frac{1}{\partial^+} \bar{\partial}_p \bar{\partial}_q + \frac{i}{\sqrt{2}} \frac{1}{\partial^+} \bar{d}_p \bar{d}_q \right\} \phi. \quad (3.60)$$

We list the conjugate relations for completeness

$$\delta_{\bar{J}} \bar{\phi} = \omega^{\rho\sigma} \bar{J}_{\rho\sigma} \bar{\phi} = i\sqrt{2} \omega^{\rho\sigma} \partial^+ \bar{\theta}_\rho \bar{\theta}_\sigma \bar{\phi}, \quad (3.61)$$

$$\delta_{\bar{J}} \bar{\phi} = \bar{\omega}_{\alpha\beta} J^{\alpha\beta} \bar{\phi} = \bar{\omega}_{\alpha\beta} \left\{ \frac{i}{\sqrt{2}} \partial^+ \theta^\alpha \theta^\beta - i\sqrt{2} \frac{1}{\partial^+} \partial^\alpha \partial^\beta + \frac{i}{\sqrt{2}} \frac{1}{\partial^+} d^\alpha d^\beta \right\} \bar{\phi}. \quad (3.62)$$

The variations of the generalized derivative and its conjugate are given by

$$\delta_J \bar{\nabla} = \bar{\omega}_{\alpha\beta} [J^{\alpha\beta}, \bar{\nabla}] = -\bar{\omega}_{\alpha\beta} \nabla^{\alpha\beta}, \quad (3.63)$$

$$\delta_J \bar{\nabla} = \omega^{\rho\sigma} [\bar{J}_{\rho\sigma}, \bar{\nabla}] = \frac{i\alpha}{2\sqrt{2}} \omega^{\rho\sigma} \bar{d}_\rho \bar{d}_\sigma \frac{\bar{\partial}}{\bar{\partial}^+}. \quad (3.64)$$

Invariance under $SO(8)$ is checked by doing a δ_J variation on the cubic vertex, including its complex conjugate.

In terms of

$$\mathcal{V} + \bar{\mathcal{V}} \equiv f_{abc} \int \left(\frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\nabla} \phi^c + \frac{1}{\partial^+} \phi^a \bar{\phi}^b \nabla \bar{\phi}^c \right), \quad (3.65)$$

explicit calculations yield

$$\delta_J \mathcal{V} = f_{abc} \bar{\omega}_{mn} \int \left(\left(\frac{\alpha}{2} - 1 \right) \frac{1}{\partial^+} \bar{\phi}^a \phi^b \partial^{\alpha\beta} \phi^c + \frac{i\alpha}{2\sqrt{2}} \frac{1}{\partial^+} \bar{\phi}^a \phi^b d^\alpha d^\beta \frac{\partial}{\partial^+} \bar{\phi}^c \right) \quad (3.66)$$

and

$$\begin{aligned} \delta_J \bar{\mathcal{V}} &= \overline{\delta_J \mathcal{V}} \\ &= f_{abc} \bar{\omega}_{\alpha\beta} \int \left(-\frac{\alpha}{2} \phi^a \frac{1}{\partial^+} \bar{\phi}^b \partial^{\alpha\beta} \phi^c + \frac{i}{\sqrt{2}} \phi^a \frac{1}{\partial^+} \bar{\phi}^b d^\alpha d^\beta \frac{\partial}{\partial^+} \bar{\phi}^c \right. \\ &\quad \left. + \frac{i\alpha}{2\sqrt{2}} \frac{1}{\partial^+} \phi^a \bar{\phi}^b d^\alpha d^\beta \frac{\partial}{\partial^+} \bar{\phi}^c \right) \end{aligned} \quad (3.67)$$

Adding the two yields

$$\begin{aligned} \delta_J (\mathcal{V} + \bar{\mathcal{V}}) &= (\alpha - 1) f_{abc} \bar{\omega} \times \\ &\quad \times \int \left(\frac{1}{\partial^+} \bar{\phi}^a \phi^b \partial^{\alpha\beta} \phi^c + \frac{i}{\sqrt{2} \partial^+} \bar{\phi}^a \phi^b d^\alpha d^\beta \frac{\partial}{\partial^+} \bar{\phi}^c \right). \end{aligned} \quad (3.68)$$

The cubic vertex is therefore $SO(8)$ invariant if $\alpha = 1$. This determines the generalized derivative to be

$$\bar{\nabla} = \bar{\partial} + \frac{i}{4\sqrt{2}\partial^+} \bar{d}_\rho \bar{d}_\sigma \partial^{\rho\sigma} . \quad (3.69)$$

This result has also been checked explicitly by performing the Grassmann integrations and looking at the components (we present some of those checks in Appendix A).

To obtain this result we have used the antisymmetry of the structure functions, the chiral constraints, the “inside-out” constraints, and performed integrations by parts on the coordinates and Grassmann variables. In particular, using the relation between the chiral field and its conjugate implied by the “inside out” constraints,

$$\bar{\phi} = \frac{1}{2 \cdot 4!} \epsilon^{\alpha\beta\rho\sigma} \bar{d}_\rho \bar{d}_\sigma \bar{d}_\alpha \bar{d}_\beta \frac{1}{\partial^{+2}} \phi , \quad (3.70)$$

we deduce two magical identities

$$f_{abc} \int \frac{1}{\partial^{+2}} \bar{\phi}^a \phi^b \bar{\partial} \phi^c = 0 , \quad (3.71)$$

$$f_{abc} \int \frac{1}{\partial^+} \bar{\phi}^a \frac{1}{\partial^+} \phi^b d^\alpha d^\beta \partial \bar{\phi}^c = 0 . \quad (3.72)$$

In this light-cone form Lorentz invariance in ten dimensions is automatic once little group invariance has been established. We have therefore shown ten-dimensional invariance since the quartic term does not need to be changed. Thus we now have a complete light-cone superspace description of ten-dimensional $\mathcal{N} = 1$ Yang-Mills. The action is simply

$$\int d^{10}x \int d^4\theta d^4\bar{\theta} \mathcal{L} , \quad (3.73)$$

where

$$\begin{aligned} \mathcal{L} = & -\bar{\phi} \frac{\square}{\partial_{+2}} \phi + \frac{4g}{3} f^{abc} \left(\frac{1}{\partial_{+}} \bar{\phi}^a \phi^b \bar{\nabla} \phi^c + \frac{1}{\partial_{+}} \phi^a \nabla \phi^b \partial \bar{\phi}^c \right) \\ & -g^2 f^{abc} f^{ade} \left(\frac{1}{\partial_{+}} (\phi^b \partial^{+} \phi^c) \frac{1}{\partial_{+}} (\bar{\phi}^d \partial^{+} \bar{\phi}^e) + \frac{1}{2} \phi^b \bar{\phi}^c \phi^d \bar{\phi}^e \right). \end{aligned} \quad (3.74)$$

The $\square$ is now the ten-dimensional d'Alembertian and the superfields have an added dependence on the six new coordinates.

CHAPTER 4

THE SUPERGRAVITY SYSTEM

Having oxidized four-dimensional Yang-Mills into its ten-dimensional parent, we now focus on the analogous situation in Supergravity. In many ways the Yang-Mills system serves as a perfect toy model for the Supergravity system. Again, our initial focus is on the maximally supersymmetric theory in four dimensions, $\mathcal{N} = 8$ Supergravity. Starting with its light-cone superspace description we show how to obtain the fully eleven-dimensional theory in superspace.

A Supergravity theory requires a fermionic companion to the usual bosonic graviton (gravitational field). This fermionic counterpart is the spin $\frac{3}{2}$ gravitino. The discovery of this particle in a laboratory would be a triumph for Supergravity because it is the only consistent field theory for interacting fields involving spin $\frac{3}{2}$.

The presence of these fermion fields does not alter the classical predictions of general relativity (fermion exchange leads to a short-range potential). However, at high energies (short distances) Supergravity is much better behaved as a quantum theory than general relativity. Infinities in the S-matrix in the first and second order quantum corrections cancel due to supersymmetry (as discussed in the Introduction). How far these cancelations persist is an open question. But this finiteness implies that Supergravity has predictive power.

4.1 Eleven-Dimensional Supergravity

Eleven dimensions is interesting because a maximal Supergravity theory in that dimension, on reduction to $d = 4$ yields particles with spin less than or equal to two. The bosonic field content of eleven-dimensional Supergravity consists of the elfbein, $e_\mu{}^a$ and a completely antisymmetric 3-form potential $A_{\mu\nu\rho}$ with field strength

$F_{\mu\nu\rho\sigma} = \partial_{[\mu} A_{\nu\rho\sigma]}$. In terms of the $SO(9)$ little group in eleven dimensions these correspond to a total of 128 bosonic states. The fermionic content consists of a single spin $\frac{3}{2}$ Majorana field, Ψ_μ which has 128 fermionic states. The $N = 1$ Supergravity action in eleven dimensions is [3]

$$\begin{aligned} L = & -\frac{1}{2\kappa^2} e e^{\rho b} e^{\sigma c} R_{\rho\sigma cb}(e, \omega) - \frac{e}{2} \bar{\Psi}_\mu \gamma^{\mu\rho\sigma} D_\rho \left(\frac{\omega + \hat{\omega}}{2} \right) \Psi_\sigma \\ & - \frac{1}{48} e F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} + \frac{2\kappa}{(12)^4} e \epsilon^{\mu_1 \dots \mu_{11}} F_{\mu_1 \dots \mu_4} F_{\mu_5 \dots \mu_8} \hat{A}_{\mu_9 \mu_{10} \mu_{11}} \\ & + \frac{\kappa}{192} e \left(\bar{\Psi}_\mu \gamma^{\mu\nu\rho\sigma\alpha\beta} \Psi_\beta + 12 \bar{\Psi}^\nu \gamma^{\rho\sigma} \Psi^\beta \right) \left(F_{\nu\rho\sigma\alpha} + \hat{F}_{\nu\rho\sigma\alpha} \right) \end{aligned}$$

The spin-connection ω is determined by the variation $\frac{\delta}{\delta \omega_{\mu ab}} L = 0$ to be

$$\begin{aligned} \omega_{\mu ab}(e) = & e_a^\rho (\partial_\nu e_{b\rho} - \partial_\rho e_{b\nu}) - e_b^\rho (\partial_\nu e_{a\rho} - \partial_\rho e_{a\nu}) + e_a^\rho e_b^\sigma (\partial_\sigma e_{c\rho} - \partial_\rho e_{c\sigma}) e_\nu^c \\ & + \frac{\kappa^2}{4} \left(\bar{\Psi}_\mu \gamma_a \Psi_b - \bar{\Psi}_\mu \gamma_b \Psi_a + \bar{\Psi}_a \gamma_\mu \Psi_b \right) - \frac{\kappa^2}{8} \bar{\Psi}^\alpha \gamma_{\alpha\mu\beta} \gamma_{ab} \Psi^\beta \end{aligned}$$

where $\mu, \nu, \dots$ run over all eleven directions, $a, b, \dots$ are locally flat elfbein coordinates (not to be confused with our notation for the transverse directions) and the curvature is defined as

$$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ac} \omega_\nu{}^c{}_b - \omega_{\nu ac} \omega_\mu{}^c{}_b \quad (4.1)$$

$\gamma^{\mu_1 \dots \mu_n}$ denotes the completely antisymmetric product of n γ matrices and the covariant derivative is defined as $D_\nu = \partial_\nu + \frac{1}{8} [\gamma^a, \gamma^b] \omega_{\nu ab}$. $\hat{\omega}$ and $\hat{F}$ are supercovariant versions of themselves.

The eleven-dimensional action is invariant under local supersymmetry transformations. These transformations may be expressed in terms of an infinitesimal space-time dependent Grassman parameter $\epsilon(x)$ which transforms as a Majorana spinor.

The transformations that leave the action (4.1) invariant are

$$\begin{aligned}
\delta e_\mu{}^a &= -\kappa \bar{\epsilon} \gamma^a \Psi_\mu \\
\delta A_{\mu\nu\rho} &= \frac{3}{2} \bar{\epsilon} \gamma_{[\mu\nu} \Psi_{\rho]} \\
\delta \Psi_\mu &= \frac{1}{\kappa} D_\mu (\hat{\omega}) \epsilon + \frac{1}{144} \left(\gamma_\mu{}^\rho \eta^{\sigma\chi} - 8 \delta_\mu{}^\rho \gamma^{\eta\sigma\chi} \right) \epsilon \hat{F}_{\rho\eta\sigma\chi}
\end{aligned} \tag{4.2}$$

Working in the 1.5 order formalism the aim is to formulate the theory in the light-cone gauge. This may be done in component form (as we will show for gravity) but due to sheer size is not a very useful approach. Mindful of our ultimate aim, we formulate the theory entirely in terms of the elfbein variables $e_\mu{}^a$. This provides us a framework in which fermions may be introduced naturally. On the light-front $\mu = \oplus, \ominus, \mathbb{1}$ where $\mathbb{1} = 1 \dots 8$. $a, b \dots$ are the locally flat light-cone indices, $a = +, -, i$ where $i = 1 \dots 8$ (again, with metric $(- + + \dots)$). We parametrize the elfbeins as follows

$$\begin{aligned}
e_\oplus{}^+ &= e^{\frac{1}{2}\phi} \\
e_\ominus{}^- &= e^{\frac{1}{2}\phi}
\end{aligned} \tag{4.3}$$

We define the symmetric object g_{ij} (in a metric-based approach, this is the transverse metric) by

$$g_{\mathbb{1}\mathbb{1}} = e_{\mathbb{1}}{}^l e_{\mathbb{1}}{}^l = e^\xi \gamma_{ij} \tag{4.4}$$

where ξ is a real field and γ_{ij} is a real symmetric unimodular matrix that satisfies

$$\gamma_{ij} \gamma^{jk} = \delta_i{}^k$$

We work in the light-cone gauge by choosing [23–28]

$$\begin{aligned} e_{\ominus}^{+} &= 0 \\ e_{\ominus}^k &= 0 \\ e_{\textcircled{1}}^{+} &= 0 \end{aligned}$$

For the final gauge condition, we choose $\phi = \frac{1}{2}\xi$.

The Einstein-Hilbert part of the Supergravity action is

$$L = -\frac{1}{2\kappa^2} e e^{\rho b} e^{\sigma c} R_{\rho\sigma cb}(e, \omega) \quad (4.5)$$

and has equations of motion, $R_{\mu\nu} = 0$. In order to work in perturbation theory we choose

$$\gamma_{ij} = (e^h)_{ij} \quad (4.6)$$

where h_{ij} is a symmetric tracefree matrix. We rescale $h = \frac{1}{\kappa}h$ to obtain

$$\gamma_{ij} = \delta_{ij} + \kappa h_{ij} + \frac{\kappa^2}{2} h_{im} h_{mj} + \mathcal{O}(\kappa^3) \quad (4.7)$$

The $R_{--} = 0$ equation of motion can be used to eliminate ξ in favour of γ

$$\xi = 0 + \mathcal{O}(\kappa^2)$$

The $R_{j-} = 0$ equation of motion gives

$$e_{\textcircled{1}}^{-} = -\kappa \frac{\partial_m}{\partial^-} h_{jm} + \mathcal{O}(\kappa^2) \quad (4.8)$$

while the $R_{+-} = 0$ equation of motion determines

$$e_{\oplus}^{-} = -\frac{\kappa}{2} \frac{\partial_i \partial_m}{\partial^{-2}} h_{im} + \mathcal{O}(\kappa^2) \quad (4.9)$$

The gravity action, $L = -\frac{1}{2\kappa^2} e e^{\rho b} e^{\sigma c} R_{\rho\sigma cb}$ may then be written entirely in terms of the physical variables h_{ij} and up to order κ reads

$$\begin{aligned} L_g^{\kappa} = & \frac{1}{4} h_{ij} \square h_{ij} + \frac{\kappa}{4} h_{ij} \frac{\partial_i \partial_j}{\partial^{-2}} (\partial^{-} h_{mn} \partial^{-} h_{mn}) + \frac{\kappa}{2} h_{mk} (\partial_i \partial^{-} h_{mk}) \frac{\partial_l}{\partial^{-}} h_{il} \\ & + \kappa h_{mk} (\partial_k \partial^{-} h_{im}) \partial_l h_{il} + \frac{\kappa}{2} h_{mk} (\partial^{-} h_{mk}) \partial_l h_{il} + \kappa (\partial^{-} h_{im}) (\partial_k h_{mk}) \partial_l h_{il} \\ & + \frac{\kappa}{4} h_{ij} (\partial_i h_{kl}) \partial_j h_{kl} - \frac{\kappa}{2} h_{mk} (\partial_l h_{il}) \partial_k h_{im} - \frac{\kappa}{2} h_{jl} (\partial_k h_{ij}) \partial_l h_{ik} \end{aligned}$$

The gravitino gets relatively more cumbersome to deal with in components. The gravitino-dependant terms in eleven-dimensional Supergravity are

$$\begin{aligned} L_{\Psi} = & -\frac{e}{2} \overline{\Psi}_{\mu} \gamma^{\mu\rho\sigma} D_{\rho} \left(\frac{\omega + \hat{\omega}}{2} \right) \Psi_{\sigma} \\ & + \frac{\kappa}{96} e \left(\overline{\Psi}_{\mu} \gamma^{\mu\nu\rho\sigma\alpha\beta} \Psi_{\beta} + 12 \overline{\Psi}^{\nu} \gamma^{\rho\sigma} \Psi^{\alpha} \right) F_{\nu\rho\sigma\alpha} \end{aligned}$$

Each spinor may be projected into an upper and lower component as defined below

$$\gamma^{+} \gamma^{-} \Psi^{\mu} = 2 \Psi^{\mu(+)} \quad ; \quad \gamma^{-} \gamma^{+} \Psi^{\mu} = 2 \Psi^{\mu(-)}$$

Lightcone gauge is chosen by setting

$$\Psi_{-} = 0 \quad (4.10)$$

We also make the additional gauge choice [29]

$$\gamma \cdot \Psi = \gamma^i \Psi^i - \gamma^+ \Psi^- = 0 \quad (4.11)$$

which implies that $\gamma^i \Psi^{i(-)} = 0$ and thus allows us to define a physical “shifted” gravitino field (with 128 components)

$$\hat{\Psi}^i(-) = \left(\delta^{it} + \frac{1}{9} \gamma^i \gamma^t \right) \Psi^{t(-)} \quad (4.12)$$

For convenience, we suppress the hat over the Ψ . The equations of motion read

$$\gamma^{\mu\nu\lambda} D_\nu(\omega) \Psi_\lambda = \frac{\kappa}{96} \gamma^{\mu\nu\rho\sigma\alpha\beta} \Psi_\beta F_{\nu\rho\sigma\alpha} + \frac{\kappa}{8} \gamma^{\rho\sigma} \Psi^\alpha F_{\rho\sigma\alpha} \quad (4.13)$$

In principle, it is straightforward to rewrite the Gravitino action in the LC_2 form (some field redefinitions are needed, even at order κ to ensure that time derivatives do not appear. Instead, they are pushed into higher order terms in the Lagrangian). However, there is little benefit in doing this since the terms that appear are so numerous that they are not very useful for analytic computations. In superspace these terms shrink to an amazing degree and become tremendously tractable.

Similar arguments apply to the 3-form field. However, we must indicate how we work in light-cone gauge. The equations of motion governing the 3-form read

$$\begin{aligned} \partial_\mu F^{\mu\nu\rho\sigma} = & -\frac{\kappa}{576} \epsilon^{\mu_1\ldots\mu_8\nu\rho\sigma} F_{\mu_1\ldots\mu_4} F_{\mu_5\ldots\mu_8} \\ & -\frac{\kappa}{96} e \partial_\alpha \left(\bar{\Psi}_\mu \gamma^{\mu\alpha\nu\rho\sigma\beta} \Psi_\beta \right) - \frac{\kappa}{8} e \partial_\alpha \left(\bar{\Psi}^\alpha \gamma^{\nu\rho} \Psi^\sigma \right) \end{aligned}$$

We choose light-cone gauge by setting

$$\begin{aligned} A_{-ij} &= A^{+ij} = 0 \\ A_{-+k} &= A^{-+k} = 0 \end{aligned} \quad (4.14)$$

Again, we will not work this out in LC_2 although it is somewhat simpler than the gravitino.

This eleven-dimensional Supergravity theory on reduction to four dimensions leads to the maximally supersymmetric $\mathcal{N} = 8$ Supergravity. This theory will be the focus of our next section.

4.2 $(\mathcal{N} = 8, d = 4)$ Supergravity in Light-Cone Superspace

Based on the fact that both the $(\mathcal{N} = 8, d = 4)$ Supergravity theory and the $(\mathcal{N} = 4, d = 4)$ SuperYang-Mills theory are maximally supersymmetric, the authors of reference [12] introduced a constrained superfield and a superspace action for the $\mathcal{N} = 8$ theory. We start with this superspace action and show that it may be used to describe the $(\mathcal{N} = 1, d = 11)$ Supergravity theory in light-cone superspace.

4.2.1 Field Content

The physical degrees of freedom of the eleven-dimensional $\mathcal{N} = 1$ Supergravity theory are classified in terms of the transverse little group $SO(9)$ with the Graviton $G^{(MN)}$ transforming as a symmetric second-rank tensor, the three-form $A^{[MNP]}$ as an anti-symmetric third-rank tensor and the Rarita-Schwinger field as a spinor-vector Ψ^M ($M, N, \dots$ are $SO(9)$ indices). This theory on reduction to four dimensions leads to the maximally supersymmetric $\mathcal{N} = 8$ theory.

The relevant decomposition is,

$$SO(9) \supset SO(2) \times SO(7) . \quad (4.15)$$

All told, after decomposition, the $\mathcal{N} = 8$ theory has a spectrum comprised of a metric, twenty-eight vector fields, seventy scalar fields, fifty-six spin one-half fields, eight spin three-half fields and their conjugates [29]. The $SO(7)$ symmetry is an internal one and can in fact be upgraded to an $SU(8)$ symmetry. However, it is important to remember that it is really the $SO(7)$ which is relevant when we “oxidize” the theory to $d = 11$. $SO(7)$ indices are represented by $m, n, \dots$

All the physical degrees of freedom of the $\mathcal{N} = 8$ theory are captured by a single complex superfield [11]

$$\begin{aligned} \phi(y) = & \frac{1}{\partial^{+2}} h(y) + i\theta^\alpha \frac{1}{\partial^{+2}} \bar{\psi}_\alpha(y) + i\theta^{\alpha\beta} \frac{1}{\partial^{+}} \bar{A}_{\alpha\beta}(y) \\ & - \theta^{\alpha\beta\gamma} \frac{1}{\partial^{+}} \bar{\chi}_{\alpha\beta\gamma}(y) - \theta^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta}(y) + i\tilde{\theta}_{\alpha\beta\gamma}^{(5)} \chi^{\alpha\beta\gamma}(y) \\ & + i\tilde{\theta}_{\alpha\beta}^{(6)} \partial^{+} A^{\alpha\beta}(y) + \tilde{\theta}_\alpha^{(7)} \partial^{+} \chi^\alpha(y) + \tilde{\theta}^{(8)} \partial^{+2} \bar{h}(y), \end{aligned} \quad (4.16)$$

where

$$\theta^{\alpha_1 \dots \alpha_n} \equiv \frac{1}{n!} \theta^{\alpha_1} \dots \theta^{\alpha_n}, \quad \tilde{\theta}_{\beta_1 \dots \beta_{8-n}}^{(n)} \equiv \frac{1}{n!} \theta^{\alpha_1 \dots \alpha_n} \epsilon_{\alpha_1 \dots \alpha_n \beta_1 \dots \beta_{8-n}}. \quad (4.17)$$

The γ -matrices allow us to translate bi-spinor indices into vector indices and vice versa. For example, the complex $\bar{A}_{\alpha\beta}$ represent

$$\theta^{\alpha\beta} \bar{A}_{\alpha\beta} = -\frac{1}{8} \theta \gamma^m \theta \bar{A}_m - \frac{1}{8} \theta \gamma^{mn} \theta \bar{A}_{[mn]}, \quad (4.18)$$

while the real $C_{\alpha\beta\gamma\delta}$ may be decomposed as

$$\begin{aligned} \theta^{\alpha\beta\gamma\delta} C_{\alpha\beta\gamma\delta} = & \frac{1}{64} \theta \gamma^m \theta \left\{ \theta \gamma^p \theta C_{(mp)} - \theta \gamma^{mp} \theta C_p - \right. \\ & \left. - \theta \gamma^{pq} \theta C_{[mpq]} \right\}. \end{aligned} \quad (4.19)$$

These tensor fields make up the two bosonic representations under the decomposition $SO(9) \subset SO(7) \times SO(2)$ of eleven-dimensional Supergravity. The $G^{(MN)}$ (**44** of $SO(9)$) split up as

$$\mathbf{44} = \mathbf{27} + \mathbf{7} + \mathbf{7} + \mathbf{1} + \mathbf{1} + \mathbf{1}. \quad (4.20)$$

$C_{(mp)}$ represents the **27** while $\bar{A}_m, A^m$ represent the $\mathbf{7} + \mathbf{7}$. Similarly the three-form $A^{[MNP]}$ (**84** of $SO(9)$) splits into

$$\mathbf{84} = \mathbf{35} + \mathbf{21} + \mathbf{21} + \mathbf{7}. \quad (4.21)$$

These correspond to $C_{[mpq]}$, $\bar{A}_{[mn]}$, $A_{[mn]}$ and C_p respectively. All fields are local in the modified light-cone coordinates

$$y = (x, \bar{x}, x^+, y^- \equiv x^- - \frac{i}{\sqrt{2}} \theta^\alpha \bar{\theta}_\alpha). \quad (4.22)$$

The superfield ϕ and its complex conjugate $\bar{\phi}$ satisfy the chiral constraints

$$d^\alpha \phi = 0; \quad \bar{d}_\alpha \bar{\phi} = 0, \quad (4.23)$$

and are related through the “inside-out” constraints

$$\phi = \frac{1}{8! \cdot 4} (d)^8 \frac{1}{\partial^{+4}} \bar{\phi}. \quad (4.24)$$

where d^8 is simply the product of all eight chiral derivatives. The $\mathcal{N} = 8$ Supergravity action to order κ is then simply

$$\int d^4x \int d^8\theta d^8\bar{\theta} \mathcal{L} \equiv \int \mathcal{L}, \quad (4.25)$$

where

$$\mathcal{L} = -\bar{\phi} \frac{\square}{\partial^4} \phi + \left(\frac{4\kappa}{3\partial^4} \bar{\phi} \bar{\partial} \bar{\partial} \phi \partial^2 \phi - \frac{4\kappa}{3\partial^4} \bar{\phi} \bar{\partial} \partial^+ \phi \bar{\partial} \partial^+ \phi + \text{c.c.} \right) \quad (4.26)$$

Grassmann integration is normalized so that $\int d^8\theta (\theta)^8 = 1$.

4.2.2 A Simpler $\mathcal{N} = 8$ Three-Point Vertex

The three-point vertex in this action, seems highly non-local and cumbersome (relative to the $\mathcal{N} = 4$ vertex). However, its form can be greatly simplified leading to a single term very similar to that in the Yang-Mills case [13]. We start by partially integrating the first term with respect to $\bar{\partial}$ to obtain

$$\int \left\{ -\frac{\bar{\partial}}{\partial^4} \bar{\phi} \bar{\partial} \phi \partial^2 \phi - \frac{1}{\partial^4} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^2 \phi \right\}. \quad (4.27)$$

The last ϕ in the first term of equation (4.27) may be rewritten as a $\bar{\phi}$, using the inside-out relation. We then partially integrate the $(d)^8$ onto the $\bar{\phi}$ and use the inside-out relation again. The second term in equation (4.27) is partially integrated with respect to ∂^+ to yield two terms.

Thus the first term in the 3-point vertex is now

$$\int \left\{ -\bar{\partial} \phi \bar{\partial} \phi \frac{1}{\partial^2} \bar{\phi} + \frac{1}{\partial^3} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^+ \phi + \frac{1}{\partial^4} \bar{\phi} \bar{\partial} \partial^+ \phi \bar{\partial} \partial^+ \phi \right\}. \quad (4.28)$$

The third term in this equation exactly cancels the second term in the original vertex.

Next, we eliminate the middle term in the equation above by recognizing that

$$I = \int \frac{1}{\partial^3} \bar{\phi} \bar{\partial} \phi \bar{\partial} \partial^+ \phi = - \int \frac{1}{\partial^2} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi - I, \quad (4.29)$$

(which follows from a partial integration of the single ∂^+ in the numerator). This allows us to set

$$I = - \int \frac{1}{2} \frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi. \quad (4.30)$$

We thus obtain a very concise three-point vertex

$$-2\kappa \int \frac{1}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \bar{\partial} \phi + c.c.. \quad (4.31)$$

This simple form allows for comparison with the $\mathcal{N} = 4$ SuperYang-Mills three-point vertex

$$\frac{4}{3} g \int f^{abc} \frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\partial} \phi^c + c.c.. \quad (4.32)$$

We will exploit this similarity in structure between the $\mathcal{N} = 8$ and $\mathcal{N} = 4$ cases when tackling the four-point vertex.

4.3 Oxidizing ($\mathcal{N} = 8, d = 4$) Supergravity to ($\mathcal{N} = 1, d = 11$) Supergravity

4.3.1 Eleven Dimensions

We now proceed to show that the fully interacting ($\mathcal{N} = 8, d = 4$) theory can be restored to its eleven-dimensional progenitor without altering the superfield. This enables us to formulate the ($\mathcal{N} = 1, d = 11$) theory without auxiliary fields in light-cone superspace.

The first step is to generalize the two transverse variables to nine. In the Yang-Mills case the compactified $SO(6)$ was easily described by $SU(4)$ parameters and we made use of the convenient bi-spinor notation. In the present case the compactified $SO(7)$ has no equivalent unitary group so we simply introduce additional real coordinates x^m and their derivatives ∂^m (where m runs from 4 through 10). The chiral

superfield remains unaltered except for the added dependence on the extra coordinates

$$h(y) = h(x, \bar{x}, x^m, y^-), \text{ etc... } . \quad (4.33)$$

These extra variables will be acted on by new operators that will restore the higher-dimensional symmetries.

4.3.2 The SuperPoincaré Algebra in 11 Dimensions

The SuperPoincaré algebra needs to be generalized from its four-dimensional version. The $SO(2)$ generators stay the same and we propose generators of the coset $SO(9)/(SO(2) \times SO(7))$ of the form

$$\begin{aligned} J^m = & -i(x \partial^m - x^m \partial) + \frac{i}{2\sqrt{2}} \partial^+ \theta^\alpha (\gamma^m)_{\alpha\beta} \theta^\beta - \frac{i}{\sqrt{2} \partial^+} \partial^\alpha (\gamma^m)_{\alpha\beta} \partial^\beta \\ & + \frac{i}{2\sqrt{2} \partial^+} d^\alpha (\gamma^m)_{\alpha\beta} d^\beta \end{aligned} \quad (4.34)$$

$$\begin{aligned} \bar{J}^n = & -i(\bar{x} \bar{\partial}^n - x^n \bar{\partial}) + \frac{i}{2\sqrt{2}} \partial^+ \bar{\theta}_\alpha (\gamma^n)^{\alpha\beta} \bar{\theta}_\beta - \frac{i}{\sqrt{2} \partial^+} \bar{\partial}_\alpha (\gamma^n)^{\alpha\beta} \bar{\partial}_\beta \\ & + \frac{i}{2\sqrt{2} \partial^+} \bar{d}_\alpha (\gamma^n)^{\alpha\beta} \bar{d}_\beta \end{aligned} \quad (4.35)$$

which satisfy the $SO(9)$ commutation relations

$$\begin{aligned} [J, J^m] &= J^m, & [J, \bar{J}^n] &= -\bar{J}^n \\ [J^{pq}, J^m] &= \delta^{pm} J^q - \delta^{qm} J^p \\ [J^m, \bar{J}^n] &= i J^{mn} + \delta^{mn} J, \end{aligned} \quad (4.36)$$

where J is the same as before, $J = j$ and the $SO(7)$ generators read

$$\begin{aligned} J^{mn} = & -i(x^m \partial^n - x^n \partial^m) + \theta^\alpha (\gamma^m)^\alpha_\beta (\gamma^n)^\beta_\sigma \bar{\partial}_\sigma \\ & + \bar{\theta}_\alpha (\gamma^m)^\alpha_\beta (\gamma^n)^\beta_\sigma \partial^\sigma - \frac{1}{\sqrt{2} \partial^+} d^\alpha (\gamma^m)^\alpha_\beta (\gamma^n)^\beta_\sigma \bar{d}_\sigma . \end{aligned} \quad (4.37)$$

The full $SO(9)$ transverse algebra is generated by J , J^{mn} , J^m and $\bar{J}^n$. All rotations are specially constructed to preserve chirality. For example

$$[J^m, \bar{d}_\alpha] = 0 ; \quad [\bar{J}^n, d^\alpha] = 0 . \quad (4.38)$$

The remaining kinematical generators do not get modified

$$J^+ = j^+ , \quad J^{+-} = j^{+-} , \quad (4.39)$$

while new kinematical generators appear

$$J^{+m} = i x^m \partial^+ ; \quad \bar{J}^{+n} = i \bar{x}^n \partial^+ . \quad (4.40)$$

We generalize the linear part of the dynamical boosts to

$$\begin{aligned} J^- = & i x \frac{\partial \bar{\partial} + \frac{1}{2} \partial^m \partial_m}{\partial^+} - i x^- \partial + i \frac{\partial}{\partial^+} \left\{ \theta^\alpha \bar{\partial}_\alpha + \frac{i}{4\sqrt{2} \partial^+} (d^\alpha \bar{d}_\alpha - \bar{d}_\alpha d^\alpha) \right\} \\ & - \frac{1}{4} \frac{\partial_m}{\partial^+} \left\{ \partial^+ \theta^\alpha (\gamma^m)_{\alpha\beta} \theta^\beta - \frac{2}{\partial^+} \partial^\alpha (\gamma^m)_{\alpha\beta} \partial^\beta + \frac{1}{\partial^+} d^\alpha (\gamma^m)_{\alpha\beta} d^\beta \right\} \end{aligned} \quad (4.41)$$

The other boosts may be obtained by using the $SO(9)/(SO(2) \times SO(7))$ rotations

$$J^{-m} = [J^-, J^m] ; \quad \bar{J}^{-n} = [\bar{J}^-, \bar{J}^n] . \quad (4.42)$$

The dynamical supersymmetries are obtained by boosting

$$\begin{aligned}
 [J^-, \bar{q}_{+\beta}] &\equiv \bar{\mathcal{Q}}_\beta = -i \frac{\partial}{\partial^+} \bar{q}_{+\beta} - \frac{i}{\sqrt{2}} (\gamma^n)_{\beta\rho} q_+^\rho \frac{\partial^n}{\partial^+}, \\
 [\bar{J}^-, q_+^\alpha] &\equiv \mathcal{Q}^\alpha = i \frac{\bar{\partial}}{\partial^+} q_+^\alpha + \frac{i}{\sqrt{2}} (\gamma^m)^{\alpha\beta} \bar{q}_{+\beta} \frac{\partial^m}{\partial^+}.
 \end{aligned} \tag{4.43}$$

They satisfy

$$\{\mathcal{Q}^\alpha, q_+^\beta\} = -(\gamma^m)^{\alpha\beta} \partial^m, \tag{4.44}$$

and the supersymmetry algebra

$$\{\mathcal{Q}^\alpha, \bar{\mathcal{Q}}_\beta\} = i\sqrt{2} \delta^\alpha_\beta \frac{1}{\partial^+} \left(\partial \bar{\partial} + \frac{1}{2} \partial^m \partial^m \right). \tag{4.45}$$

A few central charges fit nicely into this framework. The $d = 11$ theory is known to have 517 central charges (corresponding to a two form and a five form which yield 462 + 55): 7 of these may be introduced by simply replacing the seven derivatives ∂^m by c-numbers Z^m . The remaining 510 charges have an eleven-dimensional origin.

Having constructed the free $\mathcal{N} = 1$ SuperPoincaré generators in eleven dimensions we turn to building the interacting theory.

4.3.3 The Generalized Derivatives

The cubic interaction in the $\mathcal{N} = 8$ Lagrangian explicitly contains the transverse derivative operators ∂ and $\bar{\partial}$. To achieve covariance in eleven dimensions we proceed to generalize these operators as we did for $\mathcal{N} = 4$ Yang-Mills [13]. We propose the generalized derivative

$$\bar{\nabla} = \bar{\partial} + \frac{\sigma}{16} \bar{d}_\alpha (\gamma^m)^{\alpha\beta} \bar{d}_\beta \frac{\partial^m}{\partial^+}, \tag{4.46}$$

which naturally incorporates the coset derivatives ∂^m . Here σ is a parameter still to be determined. We use the coset generators to produce its rotated partner $\bar{\nabla}$ by

$$[\bar{\nabla}, J^m] \equiv \nabla^m = -i\partial^m + \frac{i\sigma}{16} \bar{d}_\alpha (\gamma^m)^{\alpha\beta} \bar{d}_\beta \frac{\partial}{\partial^+}. \quad (4.47)$$

It remains to verify that the original derivative operator is reproduced by undoing this rotation; indeed we find the required closure

$$[\nabla^m, \bar{J}^p] = \delta^{mp} \bar{\nabla}$$

The new derivative $(\bar{\nabla}, \nabla^m)$ thus transforms as a 9-vector under the little group in eleven dimensions. We note that σ is not determined by these algebraic requirements. Instead, its value will be fixed in the next section by requiring that our generalized vertex satisfy invariance requirements. We define the conjugate derivative $\bar{\nabla}$ by requiring that

$$\nabla \bar{\phi} \equiv \overline{(\bar{\nabla} \phi)}. \quad (4.48)$$

This tells us that

$$\bar{\nabla} \equiv \partial + \frac{\sigma^*}{16} d^\alpha (\gamma^n)^{\alpha\beta} d^\beta \frac{\partial^n}{\partial^+} \quad (4.49)$$

This construction is akin to that for the $\mathcal{N} = 4$ Yang-Mills theory but this time it applies to the “oxidation” of the $(\mathcal{N} = 8, d = 4)$ theory to $(\mathcal{N} = 1, d = 11)$ Supergravity. This points to remarkable algebraic similarities between the two theories, with possibly profound physical consequences. It remains to show that the simple replacement of the transverse derivatives $\partial, \bar{\partial}$ by $\nabla, \bar{\nabla}$ in the $(\mathcal{N} = 8, d = 4)$ interacting theory yields the fully covariant Lagrangian in eleven dimensions.

4.3.4 Invariance of the Action

While the covariant ($\mathcal{N} = 8$, $d = 4$) Supergravity Lagrangian is known to all orders in the gravitational coupling, its light-cone superspace expression has only been constructed to first order in κ (three-point coupling). We note that the four-point gravity vertex is indeed known in component form [27, 28] but a superfield expression remains elusive. We will work with the theory to cubic order in this section and in the next section propose a quartic interaction term for the theory. The kinetic term is trivially made $SO(9)$ -invariant by including the seven extra transverse derivatives in the d'Alembertian.

We start from our simplified version (4.31) of the $\mathcal{N} = 8$ three-point vertex. We propose that the eleven-dimensional vertex is of the same form but with the transverse derivatives replaced by the generalized derivatives introduced in the previous section:

$$\mathcal{V} = -2\kappa \int d^{11}x \int d^8\theta d^8\bar{\theta} \frac{1}{\partial^{+2}} \bar{\phi} \bar{\nabla} \phi \bar{\nabla} \phi, \quad (4.50)$$

together with its complex conjugate. To show $SO(9)$ invariance it suffices to consider the variations

$$\delta_{J^m} \phi = \frac{i}{\sqrt{2}} \omega_m \partial^+ \theta^\alpha (\gamma^m)_{\alpha\beta} \theta^\beta \phi, \quad (4.51)$$

$$\begin{aligned} \delta_{J^m} \bar{\phi} = \omega_m \left\{ \frac{i}{2\sqrt{2}} \partial^+ \theta^\alpha (\gamma^m)_{\alpha\beta} \theta^\beta - \frac{i}{\sqrt{2}\partial^+} \partial^\alpha (\gamma^m)_{\alpha\beta} \partial^\beta \right. \\ \left. + \frac{i}{2\sqrt{2}\partial^+} d^\alpha (\gamma^m)_{\alpha\beta} d^\beta \right\} \bar{\phi}, \end{aligned} \quad (4.52)$$

$$\delta_{J^m} \bar{\nabla} = -\omega_m \nabla^m, \quad (4.53)$$

where ω_m are the parameters of the $SO(9)/(SO(7) \times SO(2))$ coset transformations. The $SO(2)$ invariance is clear from the work in $d = 4$ and the $SO(7)$ invariance is covariantly realized so if we can show the invariance under the $SO(9)/(SO(7) \times SO(2))$ transformations we have shown invariance under the full $SO(9)$.

The Variation

We split the calculation into three parts based on which superfield is being varied. Terms that involve $\bar{\partial}\bar{\partial}$ or $\partial^m\partial^n$ cancel trivially. The remaining terms all involve a single $SO(2)$ derivative and a single ∂^m . We list the contributions from the variations below

Contribution from $\int \frac{1}{\partial^{+2}} \left(\delta_J \bar{\phi} \right) \bar{\nabla} \phi \bar{\nabla} \phi$:

$$\begin{aligned} \int \frac{i\sigma}{2\sqrt{2}} \left\{ \frac{8}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi - \frac{4}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi \right. \\ \left. - \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi (\gamma^m)^{\beta\kappa} (\gamma^n)^{\kappa\chi} \theta^\beta \bar{d}_\chi \partial^n \partial^+ \phi \right. \\ \left. - \frac{1}{\partial^{+3}} \bar{\phi} \theta^\beta \bar{\partial} \partial^+ \phi (\gamma^m)^{\beta\kappa} (\gamma^n)^{\kappa\chi} \bar{d}_\chi \partial^n \phi \right\} \end{aligned} \quad (4.54)$$

Contribution from $\int \frac{1}{\partial^{+2}} \bar{\phi} \left(\delta_J \bar{\nabla} \right) \phi \bar{\nabla} \phi$:

$$\int \frac{2i}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \partial^m \phi \quad (4.55)$$

Contribution from $\int \frac{1}{\partial^{+2}} \bar{\phi} \bar{\nabla} \left(\delta_J \phi \right) \bar{\nabla} \phi$:

$$\int \frac{i\sigma}{2\sqrt{2}} \left\{ \frac{4}{\partial^{+2}} \bar{\phi} \bar{\partial} \phi \partial^m \phi - \frac{1}{\partial^{+2}} \bar{\phi} \theta^\beta \bar{\partial} \phi (\gamma^m)^{\beta\kappa} (\gamma^n)^{\kappa\chi} \bar{d}_\chi \partial^n \phi \right\}, \quad (4.56)$$

These results are further simplified by use of the magical identity

$$\int \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi = \int \frac{1}{\partial^{+3}} \bar{\phi} \partial^m \phi \bar{\partial} \partial^+ \phi, \quad (4.57)$$

which follows from the duality constraint and numerous partial integrations. The final form of the variation then reads

$$\delta_J \mathcal{V} \propto \int \left(\frac{i\sigma}{\sqrt{2}} + i \right) \frac{1}{\partial^2} \bar{\phi} \bar{\partial} \phi \partial^m \phi, \quad (4.58)$$

Eleven-dimensional Poincaré invariance requires that this variation vanish. This determines σ and hence the generalized derivative

$$\bar{\nabla} = \bar{\partial} - \frac{1}{8\sqrt{2}} \bar{d}_\alpha (\gamma^m)^{\alpha\beta} \bar{d}_\beta \frac{\partial^m}{\partial^+}. \quad (4.59)$$

This completes the proof of $SO(9)$ invariance for the three-point function. It is rather remarkable that this simple replacement of the transverse derivatives renders the action covariant in eleven dimensions. In this light-cone form the Lorentz invariance in eleven dimensions is automatic once the little group invariance has been established. We have therefore proven eleven-dimensional invariance to order κ .

CHAPTER 5

DISCUSSION AND FUTURE DIRECTIONS

The fact that these reduced four-dimensional theories remember their parent versions is remarkable. It will be interesting to see if such “memory” extends to non-maximally supersymmetric theories as well. Although mathematically well defined the physical interpretation of the generalized derivative is unclear. The fact that it oxidizes a $d = 4$ gauge theory into a $d = 10$ version indicates that it relaxes the boundary conditions imposed on open strings (by thinking of the system as a D3-brane to begin with). Whenever there is a covariant derivative there is curvature at play and this may offer an interesting way of looking at our procedure. The immediate goal however revolves around determining the Supergravity action to order κ^2

5.1 Extending Gravity, to Order κ^2

We now have a superspace description of eleven-dimensional Supergravity to order κ . The obvious next step is to extend the oxidation procedure to order κ^2 . However, the ($\mathcal{N} = 8, d = 4$) four-point interaction (in light-cone superspace) is still unknown. In this section we outline a general procedure to derive the four-dimensional quartic interaction. Once this vertex is determined we expect the oxidation procedure to follow along very similar lines (thus yielding the ($\mathcal{N} = 1, d = 11$) action to order κ^2).

5.1.1 Conjectured Quartic Vertex

The similarity between the three-point functions of the $\mathcal{N} = 4$ Yang-Mills and $\mathcal{N} = 8$ Supergravity theories is quite suggestive. Based on this comparison we con-

jecture that the ($\mathcal{N} = 8, d = 4$) four-point vertex is simply

$$\mathcal{V} = \kappa^2 \int \left\{ \bar{\phi} \phi \bar{\partial} \phi \partial \bar{\phi} + \beta \frac{1}{\partial^{+2}} (\bar{\partial} \phi \partial^{+2} \phi) \frac{1}{\partial^{+2}} (\partial \bar{\phi} \partial^{+2} \bar{\phi}) \right\} \quad (5.1)$$

where β remains to be fixed. A direct check of this result can be achieved by comparison with the component form of light-cone gravity. As mentioned in the previous section the four-point function for gravity in light-cone gauge is known but there exists no expression for the quartic (or higher) vertices in terms of (LC_2)superfields in the literature. In component form it is well known that the light-cone time derivative ∂_+ , makes an appearance at every order and needs to be eliminated via field-redefinitions. This redefinition needs to be understood from a superspace point of view. We hope to return to these issues in a future publication [30]. Assuming this four-point vertex (in four dimensions) its “oxidized” eleven-dimensional version is conjectured to be

$$\mathcal{V} = \kappa^2 \int \left\{ \bar{\phi} \phi \bar{\nabla} \phi \nabla \bar{\phi} + \beta \frac{1}{\partial^{+2}} (\bar{\nabla} \phi \partial^{+2} \phi) \frac{1}{\partial^{+2}} (\nabla \bar{\phi} \partial^{+2} \bar{\phi}) \right\}. \quad (5.2)$$

It remains to be seen if this simple form reproduces the full quartic vertex of eleven-dimensional Supergravity.

5.1.2 Supersymmetry Variations

A possible way to determine the four-point vertex is to start from the known two- and three-point vertices and require invariance of the action (to order κ^2) under the non-linear dynamical SuperPoincaré transformations. Simplest among these are the dynamical supersymmetry generators already derived up to first order in the coupling constant for the ($\mathcal{N} = 8, d = 4$) theory in references [12, 31].

$$\begin{aligned} \delta_{\bar{q}_-}{}^0 \phi &= \frac{\partial}{\partial^+} \bar{q}_+ \phi \\ \delta_{\bar{q}_-}{}^\kappa \phi &= -2\kappa \frac{1}{\partial^+} \left\{ \bar{\partial} \frac{\partial}{\partial \theta} \phi \partial^{+2} \phi - \partial^+ \frac{\partial}{\partial \theta} \phi \partial^+ \bar{\partial} \phi \right\}. \end{aligned} \quad (5.3)$$

This variation, clearly preserves the chirality of the superfield it acts on. Note that we can replace the $\frac{\partial}{\partial \bar{\theta}}$ in the above expression by $\bar{d}$ or $\bar{q}_+$. This is possible because the additional $\bar{\theta} \partial^+$ piece cancels between the two terms in the variation.

To determine the four-point function, we need the dynamical supersymmetry variation at order κ^2 . We do not yet know its exact form but have narrowed it down, based on the following requirements:

- Helicity = $+\frac{5}{2}$ (based on the variations at order 0 and κ)
- Dimensions = $[L]^{\frac{1}{2}}$ (again, based on the earlier variations)
- The chirality of the superfield it acts on must not be affected
- Finally, it must preserve the “inside-out” superfield constraints.

The requirement that it leave chirality invariant is satisfied through “Chiralization”. The first three constraints offer an ansatz for the variation

$$\delta_{\bar{q}_-} \kappa^2 \phi \propto \kappa^2 \frac{1}{\partial^{+n}} \left\{ \partial^{+a} \bar{d}_{\beta} \phi \partial^{+b} \phi \partial^{+c} \partial \bar{\phi} \right\} + \dots \quad (5.4)$$

with the restriction, $a + b + c = 3 + n$ (the $\dots$ signify that more terms need to be added to ensure chirality). Once this expression is known the four-point interaction is easy to determine. In a future publication [30] we will derive the exact form of $\delta_{\bar{q}_-} \kappa^2$ and prove in addition closure of the supersymmetry algebra with the SuperPoincaré generators.

The lowest order dynamical supersymmetry variations for the ($\mathcal{N} = 1, d = 11$) theory were detailed in equation (4.43). We conjecture that at first order in coupling the variations are obtained by simply oxidizing the four-dimensional result above. That

is,

$$\delta_{\bar{Q}-} \kappa \phi = -2\kappa \frac{1}{\partial^+} \left\{ \bar{\nabla} \frac{\partial}{\partial \theta} \phi \partial^{+2} \phi - \partial^+ \frac{\partial}{\partial \theta} \phi \partial^+ \bar{\nabla} \phi \right\} \quad (5.5)$$

This result needs to be checked but we believe it to be correct since it serves as a bridge between the two- and three-point vertices.

5.1.3 Chiralization

“Chiralization” is a descent procedure whereby non-chiral expressions are rendered chiral. For any general non-chiral expression of the form $A \bar{B}$ (where A is any compound chiral function and $\bar{B}$ a compound anti-chiral function) we define a “chiral product” through a descent relation in chiral derivatives

$$\mathcal{C}(A \bar{B}) = A \bar{B} + \sum_{n=1}^8 \frac{(-1)^n}{n!} \frac{\bar{d}_{\alpha_1 \dots \alpha_n}}{(-i\sqrt{2}\partial^+)^n} (A d^{\alpha_n \dots \alpha_1} \bar{B}), \quad (5.6)$$

where $\bar{d}_{\alpha_1 \dots \alpha_n} = \bar{d}_{\alpha_1} \dots \bar{d}_{\alpha_n}$ and $d^{\alpha_n \dots \alpha_1} = d^{\alpha_n} \dots d^{\alpha_1}$.

$\mathcal{C}(A \bar{B})$ is now a chiral function and satisfies $d\mathcal{C} = 0$. Clearly the descent series involves as many terms as there are supersymmetries in the theory.

This procedure is equally applicable to other non-chiral forms. For example the product $\bar{d}AB$ where both A and B are chiral functions is chiralized by the addition of the term

$$-\partial^+ A \frac{\bar{d}}{\partial^+} B \quad (5.7)$$

Similarly the addition of the two terms

$$-\partial^+ \bar{d} A \frac{\bar{d}}{\partial^+} B + \partial^{+2} A \frac{\bar{d}\bar{d}}{\partial^{+2}} B \quad (5.8)$$

to the expression $\bar{d}\bar{d}AB$, chiralizes it and so on.

This procedure is invaluable when dealing with variations (with respect to supersymmetry and the boosts) at higher orders. This simple recipe ensures that all variations respect the superfield chirality structure.

5.2 Dual Descriptions

Despite our focus on formulating supersymmetric theories in light-cone superspace, the ultimate goal (as detailed in the introduction) is to understand M-Theory. This thesis studies one possible approach to understanding M-Theory through its low-energy limit eleven-dimensional Supergravity. There are other venues that also look promising.

M-Theory casts well defined shadows on lower dimensions. From the AdS/CFT [32] point of view conformally invariant gauge theories are of great interest (the rank of the gauge group is related to flux in the dual description). The $\mathcal{N} = 4$ SuperYang-Mills theory is dual to Type IIB String Theory compactified on $AdS_5 \times S^5$. Similarly M-Theory when compactified on $AdS_4 \times S^7$ is described in terms of a three-dimensional superconformal theory (governed by the symmetry group $OSp(8|4)$). This theory has not yet been constructed and will be a very interesting thing to tackle in this light-cone superspace framework. The $OSp(8|4)$ is trivially realized on the superfields [33] using our formalism and this is a good indication that we may be able to write down a Lagrangian for this theory. A better understanding of these dual theories will certainly teach us something about the structure of M-Theory.

APPENDIX A YANG-MILLS

A.1 Duality Theorems

Consequence #1 ;

$$f_{abc} \int \frac{1}{\partial^{+2}} \bar{\phi}^a \phi^b \bar{\partial} \phi^c = 0 \quad (\text{A.1})$$

$$\begin{aligned} \text{Proof: } \int \frac{1}{\partial^{+2}} \bar{\phi}^a \phi^b \bar{\partial} \phi^c &= \int \frac{1}{\partial^{+2}} \bar{\phi}^a \frac{3d^4}{\partial^{+2}} \bar{\phi}^b \bar{\partial} \phi^c \\ &= \int \frac{3d^4}{\partial^{+2}} \bar{\phi}^a \frac{1}{\partial^{+2}} \bar{\phi}^b \bar{\partial} \phi^c = \int \phi^a \frac{1}{\partial^{+2}} \bar{\phi}^b \bar{\partial} \phi^c \end{aligned}$$

which is symmetric in a and b .

Consequence #2 ;

$$f_{abc} \int \frac{1}{\partial^{+}} \bar{\phi}^a \frac{1}{\partial^{+}} \phi^b d^\alpha d^\beta \partial \bar{\phi}^c = 0 \quad (\text{A.2})$$

$$\begin{aligned} \text{Proof: } \int \frac{1}{\partial^{+}} \bar{\phi}^a \frac{1}{\partial^{+}} \phi^b d^\alpha d^\beta \partial \bar{\phi}^c &= \int \frac{1}{\partial^{+}} d^\alpha d^\beta \bar{\phi}^a \frac{1}{\partial^{+}} \phi^b \partial \bar{\phi}^c \\ &= \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \int \frac{1}{\partial^{+}} \bar{d}_\rho \bar{d}_\sigma \phi^a \frac{1}{\partial^{+}} \phi^b \partial \bar{\phi}^c = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \int \frac{1}{\partial^{+}} \phi^a \frac{1}{\partial^{+}} \bar{d}_\rho \bar{d}_\sigma \phi^b \partial \bar{\phi}^c \end{aligned}$$

again symmetric in a and b .

A.2 Component Check

We start with the Taylor expanded version of the bosonic superfield

$$\begin{aligned}\phi(y) = & \frac{1}{\partial^+} A(x) + \frac{i}{\sqrt{2}} \theta \bar{\theta} A(x) - \frac{1}{4} (\theta \bar{\theta})^2 \partial^+ A(x) - \frac{i}{12\sqrt{2}} (\theta \bar{\theta})^3 \partial^{+2} A(x) \\ & + \frac{1}{96} (\theta \bar{\theta})^4 \partial^{+3} A(x) + 2\theta^4 \partial^+ \bar{A}(x) + \frac{i}{\sqrt{2}} \theta^\alpha \theta^\beta \bar{C}_{\alpha\beta}(x) - \frac{1}{2} \theta^\alpha \theta^\beta (\theta \bar{\theta}) \partial^+ \bar{C}_{\alpha\beta}(x) \\ & - \frac{i}{4\sqrt{2}} \theta^\alpha \theta^\beta (\theta \bar{\theta})^2 \partial^{+2} \bar{C}_{\alpha\beta}(x)\end{aligned}\quad (\text{A.3})$$

The 3-point coupling in the ten-dimensional theory is obtained by starting with the four-dimensional result and simply ‘extending’ the superspace derivative

$$\begin{aligned}S_3 = & \int d^{10}x d^4\theta d^4\bar{\theta} \alpha f^{abc} \left[\frac{1}{\partial^+} \bar{\phi}^a \phi^b \bar{\nabla} \phi^c + \text{complex conjugate} \right] \\ = & \int d^{10}x \alpha f^{abc} \left[V_{AAA} + V_{CCC} + V_{AAC} + V_{CCA} + \text{complex conjugate} \right]\end{aligned}\quad (\text{A.4})$$

where the generalized derivative is assumed to be

$$\bar{\nabla} = \bar{\partial} + B \bar{d}_\alpha \bar{d}_\beta \frac{\partial^{\alpha\beta}}{\partial^+} \quad (\text{A.5})$$

The cubic terms in A simplify to

$$\begin{aligned}V_{AAA} = & -12 \partial^+ \bar{A} A \frac{\bar{\partial}}{\partial^+} A \\ = & +12 \bar{A} A \bar{\partial} A + 12 \bar{A} \partial^+ A \frac{\bar{\partial}}{\partial^+} A\end{aligned}\quad (\text{A.6})$$

(where we have suppressed the gauge indices). To study this in component form we choose a typical term

$$\text{Typical } A^3 \text{ Term} = 3 \left\{ A_2 A_1 \partial_1 A_2 + A_2 \partial^+ A_2 \frac{\partial_1}{\partial^+} A_1 \right\} \quad (\text{A.7})$$

The cubic terms in C simplify to

$$\begin{aligned} V_{CCC} = & i 2 \sqrt{2} B \partial^+ \bar{C}_{\alpha\beta} C^{\alpha\beta} \frac{\partial^{\rho\sigma}}{\partial^+} \bar{C}_{\rho\sigma} + i 8 \sqrt{2} B \bar{C}_{\alpha\gamma} C^{\alpha\delta} \partial^{\rho\gamma} \bar{C}_{\rho\delta} \\ & + i 2 \sqrt{2} B \frac{1}{\partial^+} \bar{C}_{\alpha\beta} C^{\rho\sigma} \partial^{\alpha\beta} \partial^+ \bar{C}_{\rho\sigma} \end{aligned} \quad (\text{A.8})$$

Which in component form reads

$$\begin{aligned} V_{CCC} = & i 8 \sqrt{2} B \partial^+ A_i A_i \frac{\partial_j}{\partial^+} A_j - i 16 \sqrt{2} B A_i A_j \partial_j A_i \\ & + i 8 \sqrt{2} B \frac{1}{\partial^+} A_i A_j \partial_i \partial^+ A_j \end{aligned} \quad (\text{A.9})$$

We work with a typical term again choosing the term involving one ∂_6 , one A_6 and two A_7 's. The first two terms then give

$$i 8 \sqrt{2} B \partial^+ A_7 A_7 \frac{\partial_6}{\partial^+} A_6 - i 16 \sqrt{2} B A_7 A_6 \partial_6 A_7 \quad (\text{A.10})$$

while the third term is $T = i 8 \sqrt{2} B \frac{1}{\partial^+} A_6 A_7 \partial_6 \partial^+ A_7$. We partially integrate this to obtain

$$\begin{aligned} T = & i 8 \sqrt{2} B \partial_6 \partial^+ \left[\frac{1}{\partial^+} A_6 A_7 \right] A_7 \\ = & i 8 \sqrt{2} B \frac{\partial_6}{\partial^+} A_6 \partial^+ A_7 A_7 + i 8 \sqrt{2} B A_6 \partial_6 A_7 A_7 - T \end{aligned} \quad (\text{A.11})$$

This lets us solve for T . (terms of the form $A_7^a A_7^b (\dots)$ vanish due to the antisymmetric structure constant). The final expression then reads

$$\text{Typical } C^3 \text{ Term} = -i 12 \sqrt{2} B \left\{ A_7 A_6 \partial_6 A_7 + A_7 \partial^+ A_7 \frac{\partial_6}{\partial^+} A_6 \right\} \quad (\text{A.12})$$

A comparison with the typical A^3 term determines

$$B = \frac{i}{4\sqrt{2}} \quad (\text{A.13})$$

which is exactly what we obtain when working directly in superspace.

A.3 Useful Results

The extended derivative $\overline{\nabla}$ is defined by

$$\overline{\nabla} \phi = \overline{\partial} \phi + \frac{i\alpha}{4\sqrt{2}} \overline{d}_\alpha \overline{d}_\beta \frac{\partial^{\alpha\beta}}{\partial^+} \phi \quad (\text{A.14})$$

$$[\partial^\alpha, \overline{\nabla}] = \frac{\alpha}{4} \overline{d}_\beta \partial^{\beta\alpha} \quad ; \quad [\overline{\nabla}, \theta^\alpha] = \frac{i\alpha}{2\sqrt{2}} \overline{d}_\beta \frac{\partial^{\beta\alpha}}{\partial^+} \quad (\text{A.15})$$

$$\overline{\omega}_{\alpha\beta} [\partial^\alpha \partial^\beta, \overline{\nabla}] \phi = \frac{i\alpha}{4\sqrt{2}} \partial^+ \overline{\omega}_{\alpha\beta} [3\partial^{\alpha\beta} - 2\theta^\alpha \overline{d}_\gamma \partial^{\gamma\beta}] \phi \quad (\text{A.16})$$

$$\overline{\omega}_{\alpha\beta} [\overline{\nabla}, \theta^\alpha \theta^\beta] \phi = -\frac{i\alpha}{2\sqrt{2}} \frac{1}{\partial^+} \overline{\omega}_{\alpha\beta} [\partial^{\alpha\beta} - 2\theta^\alpha \overline{d}_\gamma \partial^{\gamma\beta}] \phi \quad (\text{A.17})$$

$$[d^\alpha d^\beta, \overline{\nabla}] \phi = \frac{i}{\sqrt{2}} \partial^+ \partial^{\alpha\beta} \phi \quad (\text{A.18})$$

$$[\overline{\nabla}, L^{\alpha\beta}] = \partial^{\alpha\beta} - \frac{i\alpha}{4\sqrt{2}} \epsilon^{\rho\sigma\alpha\beta} \overline{d}_\rho \overline{d}_\sigma \frac{\partial}{\partial^+}$$

APPENDIX B SUPERGRAVITY

B.1 Duality Theorem

Consequence #1 ;

$$\begin{aligned}
 I &= \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi \\
 &= \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \frac{d^8}{\partial^{+3}} \bar{\phi} \\
 &= \partial^+ \phi \bar{\partial} \phi \frac{\partial^m}{\partial^{+3}} \bar{\phi} \\
 &= - \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \partial^m \phi \partial^+ \phi - I
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 2I &= - \frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \partial^m \phi \partial^+ \phi \\
 &= + \frac{1}{\partial^{+3}} \bar{\phi} \partial^m \phi \bar{\partial} \partial^+ \phi + \frac{\bar{\partial}}{\partial^{+3}} \bar{\phi} \partial^m \phi \partial^+ \phi
 \end{aligned} \tag{B.2}$$

We work with the second term to obtain

$$\begin{aligned}
 \frac{\bar{\partial}}{\partial^{+3}} \bar{\phi} \partial^m \phi \partial^+ \phi &= \frac{\bar{\partial}}{\partial^{+3}} \bar{\phi} \partial^m \phi \frac{1}{\partial^{+3}} d^8 \bar{\phi} \\
 &= \frac{1}{\partial^{+3}} \bar{\phi} \partial^m \phi \bar{\partial} \partial^+ \phi
 \end{aligned} \tag{B.3}$$

which adds to the first term and leaves us with the surprising result

$$\frac{1}{\partial^{+3}} \bar{\phi} \bar{\partial} \phi \partial^m \partial^+ \phi = \frac{1}{\partial^{+3}} \bar{\phi} \partial^m \phi \bar{\partial} \partial^+ \phi \tag{B.4}$$

B.2 Useful Results

$$d^\alpha = -\frac{\partial}{\partial \bar{\theta}_\alpha} - \frac{i}{\sqrt{2}} \theta^\alpha \partial^+ = -\partial^\alpha - \frac{i}{\sqrt{2}} \theta^\alpha \partial^+ \quad (\text{B.5})$$

$$\bar{d}_\beta = \frac{\partial}{\partial \theta^\beta} + \frac{i}{\sqrt{2}} \bar{\theta}_\beta \partial^+ = \bar{\partial}_\beta + \frac{i}{\sqrt{2}} \bar{\theta}_\beta \partial^+ \quad (\text{B.6})$$

$$\partial^\alpha \phi = -\frac{i}{\sqrt{2}} \theta^\alpha \partial^+ \phi \quad ; \quad \partial^\alpha \partial^\beta \phi = -\frac{1}{2} \theta^\alpha \theta^\beta \partial^{+2} \phi \quad (\text{B.7})$$

$$[\partial^\beta, \bar{\nabla}] = \frac{i\sigma}{8\sqrt{2}} (\gamma^n)^{\beta\kappa} \bar{d}_\kappa \partial^n \quad ; \quad [\bar{\nabla}, \theta^\beta] = -\frac{\sigma}{8} (\gamma^n)^{\beta\kappa} \bar{d}_\kappa \frac{\partial^n}{\partial^+} \quad (\text{B.8})$$

$$[d^\alpha d^\beta, \bar{\nabla}] = -\frac{\sigma}{4} (\gamma^n)^{\beta\alpha} \partial^n \partial^+ - \frac{i\sigma}{2\sqrt{2}} \bar{d}_\kappa d^\beta (\gamma^n)^{\alpha\kappa} \partial^n \quad (\text{B.9})$$

$$[\partial^\alpha \partial^\beta, \bar{\nabla}] = -\frac{\sigma}{16} (\gamma^n)^{\beta\alpha} \partial^n \partial^+ + \frac{i\sigma}{4\sqrt{2}} \bar{d}_\kappa \partial^\beta (\gamma^n)^{\alpha\kappa} \partial^n \quad (\text{B.10})$$

$$[\partial^\alpha \partial^\beta, \bar{\nabla}] \phi = -\frac{3\sigma}{16} (\gamma^n)^{\beta\alpha} \partial^n \partial^+ \phi - \frac{\sigma}{8} \theta^\beta \bar{d}_\kappa (\gamma^n)^{\alpha\kappa} \partial^n \partial^+ \phi \quad (\text{B.11})$$

$$[\bar{\nabla}, \theta^\alpha \theta^\beta] = \frac{\sigma}{8} (\gamma^n)^{\beta\alpha} \frac{\partial^n}{\partial^+} + \frac{\sigma}{4} \theta^\beta \bar{d}_\kappa (\gamma^n)^{\alpha\kappa} \frac{\partial^n}{\partial^+} \quad (\text{B.12})$$

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